[redacted]

ANSWER ALL SIX QUESTIONS

1. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers with $x_n \xrightarrow[n\to+\infty]{} L$, where $L > 0$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+x_n}}$$

converges. **[10 marks]**

2.

(a) Let $a, b \in \mathbb{R}$, $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and twice differentiable in (a, b). Let A and B be the points with coordinates $(a, f(a))$ and $(b, f(b))$ respectively.

If the line segment with endpoints A and B intersects the graph of f at a point P with $P \neq A, B$ (see figure 1), prove that there exists a real number c in the interval (a, b) such that $f''(c) = 0$. **[10 marks]**

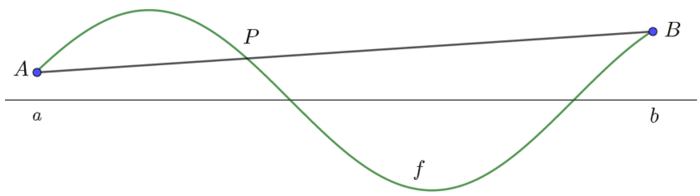


Figure 1: Plot for Question 2a.

(b) Let $a = -4$, $b = 1$ and $f(x) = \frac{1}{|x|}$. The points $A(-4, \frac{1}{4})$ and $B(1, 1)$ are on the graph of f and the line segment with endpoints A and B intersects the graph at a third point P (see figure 2). However, there is no point c in the interval $(-4, 1)$ such that $f''(c) = 0$. (You are <u>not</u> asked to prove that AB intersects the graph, nor that f'' doesn't vanish).

Explain why this doesn't violate the result of part (a). **[5 marks]**

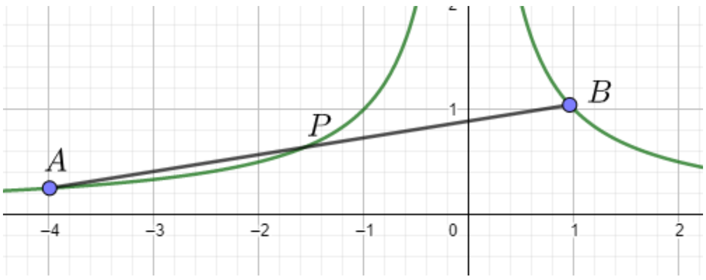


Figure 2: Plot for Question 2b.

3.

(a) Prove, by verifying the ε-δ property (Ross 17.2), that $f(x) = \dfrac{x^2}{5x-9}$ is continuous at $x_0 = 2$. **[10 marks]**

(b) Let $f : [-1,1] \to [-1,1]$ be continuous. Recall that by the Intermediate Value Theorem, there exists at least one $x \in [-1,1]$ such that $f(x) = x$. (You are not asked to prove this.)

(i) Prove that if in addition to continuity we have $|f(a) - f(b)| < |a - b|$ for all $a, b \in [-1,1]$, $a \neq b$ then there exists a *unique* $x \in [-1,1]$ such that $f(x) = x$.

(ii) In the setting of part (b)(i) either prove that f is monotone or give a carefully justified example of such a function f which is not monotone.

[15 marks]

4. Let $G = GL(2,\mathbb{R})$ be the group of 2×2 invertible matrices under the operation of matrix multiplication, and consider the subset $H \subset G$ of all matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$, with $x \in \mathbb{R}$.

(a) Show that H is a subgroup of G. **[5 marks]**

(b) What familiar group is H isomorphic to? Show that H is indeed isomorphic to that group. **[5 marks]**

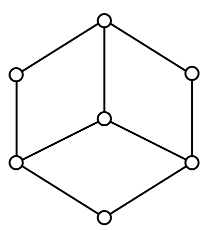
5. In each of the following cases state whether the given statement is TRUE or FALSE and provide a justification, including specific counterexamples where appropriate.

(a) The function $f : S_3 \to \mathbb{Z}_3$ defined by $f(\tau) = o(\tau) - 1$ is a group homomorphism. Here, $o(\tau)$ represents the order of the permutation τ. **[5 marks]**

(b) The groups $\mathbb{Z}_8 \times \mathbb{Z}_{10}$ and $\mathbb{Z}_{40} \times \mathbb{Z}_2$ are isomorphic. **[5 marks]**

(c) All proper subgroups of D_{13} are cyclic. (Recall that a proper subgroup is a subgroup that is not equal to the whole group.) **[5 marks]**

6. Consider the following graph:



Let G denote its group of symmetries. Justify all your answers to the questions below.

(a) How many symmetries does the graph have? **[5 marks]**

(b) Identify the symmetry group G (in terms of groups you have encountered in FPM). **[4 marks]**

(c) Let X denote the set of nine *edges* of the graph.

Is the action of G on X faithful? **[3 marks]**

Is the action of G on X transitive? **[3 marks]**

(d) Suppose we want to colour the *edges* of the graph with n colours. We consider two colourings the same if they are related by a symmetry of the graph.

How many different colourings of the edges are there? **[10 marks]**

Express your answer as a polynomial in n.