

② Let P_n be polynomials of degree n or less in x .

② Show that P_2 is a subspace of P_3 .

→ If P_2 is a subspace of P_3 must meet two conditions.

⊛ For all $X_1, X_2 \in P_2$ then $X_1 + X_2 \in P_2$.

Let X_1, X_2 be elements of P_2 such that

$$\begin{aligned} X_1 &= ax^2 + bx + c \\ X_2 &= dx^2 + ex + f \end{aligned} \quad \left| \text{ where } a, b, c, d, e \text{ and } f \text{ are constants.} \right.$$

Then

$$\begin{aligned} X_1 + X_2 &= ax^2 + bx + c + dx^2 + ex + f \\ X_1 + X_2 &= (a+d)x^2 + (b+e)x + c+f. \end{aligned} \quad \left| \text{ This is a polynomial of degree 2.} \right.$$

Therefore $X_1 + X_2 \in P_2$

⊛⊛ For all $X_1 \in P_2$, $KX_1 \in P_2$ with $K \in \mathbb{R}$.

Let X_1 be an element of P_2 such that

$$X_1 = ax^2 + bx + c, \quad \text{where } a, b, c \text{ are constants.}$$

So $KX_1 = K(ax^2 + bx + c)$ with $K \in \mathbb{R}$

$$KX_1 = Kax^2 + Kbx + cK \quad \left| \text{ This is a polynomial of degree 2.} \right.$$

Then $KX_1 \in P_2$.

Therefore P_2 is a subspace of P_3 .

⑥ Let U be the subset of P_n comprising all polynomials such that $p(1) + p'(1) = 0$ and $p(2) = 0$. Is U a subspace?

Taking

$$p(x) = \sum_{i=0}^n a_i x^i$$

We have

$$p(1) = \sum_{i=0}^n a_i$$

$$p(2) = \sum_{i=0}^n a_i 2^i$$

$$p'(x) = \sum_{i=1}^n i a_i x^{i-1} \rightarrow p'(1) = \sum_{i=1}^n i a_i$$

Then

$$p(1) + p'(1) = 0 \rightarrow \sum_{i=0}^n a_i + \sum_{i=1}^n i a_i = 0$$

$$a_0 + \sum_{i=1}^n (1+i) a_i = 0 \rightarrow a_0 = -\sum_{i=1}^n (1+i) a_i$$

And

$$p(2) = 0 \rightarrow \sum_{i=0}^n a_i 2^i = 0$$

$$a_0 + \sum_{i=1}^n a_i 2^i = 0 \rightarrow a_0 = -\sum_{i=1}^n a_i 2^i$$

Then

$$a_0 = a_0$$

$$-\sum_{i=1}^n (1+i) a_i = -\sum_{i=1}^n a_i 2^i \rightarrow 2^i = 1+i \rightarrow \begin{cases} i=0 \\ \text{or} \\ i=1 \end{cases}$$

It means U is the subset of all polynomials of degree 1. Which is a subspace of P_n , $n \geq 1$.

If we change the condition to $p(1) + p'(1) = 0$ or $p(2) = 0$. U is not a subspace, because if p meets $p(1) + p'(1) = 0$ and q meets $q(2) = 0$, there is no guarantee that $p+q$ meets $(p+q)(1) + (p+q)'(1) = 0$ or $(p+q)(2) = 0$.

③ Consider $U = \{ f \in C[a, b] \mid \int_a^b f(x) dx = 0 \}$. Determine whether U is a subspace of $C[a, b]$.

→ If U is a subspace of $C[a, b]$ must meet two conditions:

(*) For all $f, g \in U$, then $f + g \in U$.

Let f, g be functions of U . Then

$$\int_a^b f(x) dx = 0 \quad \text{and} \quad \int_a^b g(x) dx = 0.$$

$$\text{So, } \int_a^b (f+g)(x) dx = \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

$$\int_a^b (f+g)(x) dx = 0 + 0$$

$$\int_a^b (f+g)(x) dx = 0$$

By definition of U , $f+g \in U$.

(*) (*) For all $f \in U$, $kf \in U$ with $k \in \mathbb{R}$.

Let f be a function of U . Let k be a constant. So.

$$\int_a^b (kf)(x) dx = k \int_a^b f(x) dx$$

$$\int_a^b (kf)(x) dx = 0$$

By definition of U , $kf \in U$.

Then U is a subspace of $C[a, b]$

(ii) Consider $U = \{x = (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$. Determine whether U is a subspace of $\mathbb{R}^2$.

$\rightarrow U$ is not a subspace of $\mathbb{R}^2$, because if $x \in U$.

with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $x_1 x_2 = 0$, then $x_1 = 0$ or $x_2 = 0$.

So, for $u, v \in U$ we have

$$u + v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

where $u_1 = 0$ or $u_2 = 0$
and $v_1 = 0$ or $v_2 = 0$.

Without loss of generality we take $u_1 = 0, u_2 \neq 0$
 $v_1 = 0, v_2 \neq 0$.

Then

$$u + v = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

S. $u_2 v_2 \neq 0$, because $u_2 \neq 0$ and $v_2 \neq 0$.

Then $u + v \notin U$ and U is not a subspace of $\mathbb{R}^2$.

5

(i) Consider the sets $S_1 = \{(1 \ 1 \ 2)^T, (2 \ -1 \ 1)^T\}$ and

$S_2 = \{(1 \ -2 \ -1)^T, (0 \ -3 \ -3)^T\}$. Show that $\text{Span}(S_1) = \text{Span}(S_2)$

* $\text{Span}(S_1) \subseteq \text{Span}(S_2)$

Let $\vec{x}$ be an element of $\text{span}(S_1)$, then

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Considering

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = a \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \quad \left| \begin{array}{l} \text{It means } \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \in \text{Span}(S_2) \\ \text{if } a, b \neq 0. \end{array} \right.$$

Then

$$\begin{array}{lcl} 1 = a & & \\ 1 = -2a - 3b & \rightarrow & 1 = -2 - 3b \rightarrow \underline{b = -1} \\ 2 = -a - 3b & & 2 = -1 - 3b \end{array}$$

So $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$

On the other hand.

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = c \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + d \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \quad \left| \begin{array}{l} \text{It means } \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \in \text{Span}(S_2) \\ \text{if } c, d \neq 0. \end{array} \right.$$

Then

$$\begin{array}{lcl} 2 = c & & \\ -1 = -2c - 3d & \rightarrow & -1 = -4 - 3d \rightarrow d = -1 \\ 1 = -c - 3d & & 1 = -2 - 3d \end{array}$$

Then $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$

Sol.
$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \left(\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \right) + \beta \left(2 \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = (\alpha + 2\beta) \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} - (\alpha + \beta) \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}, \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in \text{span}(S_1)$$

Then $\text{span}(S_1) \subseteq \text{span}(S_2)$.

* $\text{span}(S_2) \subseteq \text{span}(S_1)$.

Let $\vec{x}$ be a element of $\text{span}(S_2)$, then.

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + s \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$$

Considering
$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \Bigg| \quad \text{It means } \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \in \text{span}(S_1)$$

if $a, b \neq 0$.

Then

$$\begin{array}{lcl} 1 & = & a + 2b \\ -2 & = & a - b \\ -1 & = & 2a + b \end{array} \rightarrow \begin{array}{l} 1 = a + 2b \\ -(-2 = a - b) \\ \hline 1 - (-2) = a - a + 2b - (-b) \\ 3 = 3b \rightarrow \underline{b=1} \rightarrow a = -1 \end{array}$$

S.
$$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

On the other hand.

$$\begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

S.

$$0 = c + 2b$$

$$-3 = c - b$$

$$-3 = 2c + b$$

$$\begin{aligned} 0 &= c + 2b \\ \rightarrow -(-3 &= c - b) \\ 3 &= 3b \end{aligned}$$

$$\rightarrow b = 1, \underline{c = -2}$$

Then

$$\begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = r \left(-\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right) + s \left(-2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -(r + 2s) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + (s + s) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{It means } \begin{pmatrix} x \\ y \end{pmatrix} \in \text{span}(S_1)$$

Then $\text{span}(S_2) \subseteq \text{span}(S_1) \quad \therefore \text{span}(S_1) = \text{span}(S_1).$

(ii) Determine whether or not the set of polynomials $S = \{1-x, 2x+x^2, 1+x+x^2\}$ is linearly independent.

Taking

$$\alpha(1-x) + \beta(2x+x^2) + \gamma(1+x+x^2) = 0$$

$$\alpha - \alpha x + 2\beta x + \beta x^2 + \gamma + \gamma x + \gamma x^2 = 0$$

$$x^2(\beta + \gamma) + x(-\alpha + 2\beta + \gamma) + \alpha + \gamma = 0$$

S.

$$\begin{aligned} \beta + \gamma &= 0 \rightarrow \gamma = -\beta \\ -\alpha + 2\beta + \gamma &= 0 \rightarrow \alpha = \beta \\ \alpha + \gamma &= 0 \rightarrow \gamma = -\alpha \end{aligned} \quad \Rightarrow \quad \underline{\alpha = \beta}$$

Then $-\alpha + 2\beta + \gamma = 0 \rightarrow \gamma - 2\gamma + \gamma = 0.$

As $\alpha, \beta, \gamma \neq 0$. the set S is not linearly independent.