

Problem 2.

1. Find the Probability that $X \leq Y$.

$P(X \leq Y)$

Soln.

Let X and Y be two independent, Exponential, distributed random Variables with parameters λ and μ resp.

$$\therefore X \sim \text{exp}(\lambda), \text{ } Y \sim \text{exp}(\mu)$$

$$P(X \leq Y) = \int_0^{\infty} \int_x^{\infty} f_{X,Y}(x,y) dy dx.$$

$$= (\lambda \cdot e^{-\lambda x}) (\mu e^{-\mu y}) \left[\because X \text{ and } Y \text{ are Independent} \right]$$
$$f(x,y) = f(x) \cdot f(y)$$

$$P(X \leq Y) = \int_0^{\infty} \int_x^{\infty} \lambda \mu e^{-\lambda x} \cdot e^{-\mu y} dy dx$$

$$= \left(\lambda \mu \int_0^{\infty} e^{-\lambda x} \left(\frac{e^{-\mu y}}{-\mu} \right) \right)_x^{\infty}$$

$$= \lambda \cdot \int_0^{\infty} e^{-(\lambda+\mu)x} dx$$

$$= \lambda \cdot \frac{e^{-(\lambda+\mu)x}}{-(\lambda+\mu)} \Bigg|_0^{\infty}$$

$$= \frac{\lambda}{\lambda + \mu}$$

2. let $Z = 1/(1+X)$ for $0 < Z < 1$.

$$f_Z(z) =$$

$$\text{let } Z = \frac{1}{1+X} \cdot \text{cdf of } Z = P(Z \leq z)$$

$$= P\left(\frac{1}{1+X} \leq z\right)$$

$$= P(1+X \geq \frac{1}{z})$$

$$\int_{\frac{1}{z}}^{\infty} \lambda \cdot e^{-\lambda u} du$$

$$= \lambda \cdot \frac{e^{-\lambda u}}{-\lambda} \Big|_{\frac{1}{z}}^{\infty}$$

$$= e^{-\lambda(z-1)}$$

for range.

range of $Z \in (0, 1)$

$$\text{Pdf of } Z = \frac{d}{dz}(\text{cdf of } Z)$$

$$= \frac{d}{dz}(e^{-(z-1)\lambda})$$

$$= -\lambda \cdot e^{-\lambda(z-1)}$$

$$\therefore f_Z(z) =$$

$$\therefore f_Z(z) = \begin{cases} -\lambda \cdot \exp(-\lambda(z-1)) & 0 < z < 1 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4.
1. Given an Expression for $f_X(x)$.
Solve for $x > 0$

$f_X(x)$

Soln.

Given: Let 'S' be a discrete random variable that takes the value 1 with probability $p \in (0, 1)$ and the value -1 with probability $(1-p)$. Let 'X' be a continuous random variable.

Let $p \in (0, 1)$

$$f_{X|S=1}(x) = \alpha \cdot e^{-\alpha x}; \alpha > 0, x > 0$$

$$f_{X|S=-1}(x) = \beta \cdot e^{\beta x}; \beta > 0, x < 0$$

(i) $\therefore$ The joint Pdf of X, S given by
for $x > 0$,

$$f_{X,S=1}(x) = p\alpha \cdot e^{-\alpha x}; \alpha > 0, x > 0$$

for $x < 0$,

$$f_{X,S=-1}(x) = (1-p)\beta \cdot e^{\beta x}; \beta > 0, x < 0$$

(ii) If $Z = |X|$, then pdf of $Z = |X|$ is
$$\begin{cases} Z = X, X > 0 \\ Z = -X, X < 0 \end{cases}$$

$$f_Z(z) = p\alpha \cdot e^{-\alpha z} + (1-p)\beta \cdot e^{-\beta z}; \alpha, \beta > 0, z \geq 0$$

The conditional distribution function is given by,

$$f_{S=1|Z}(z) = \frac{f_{Z,S=1}(z)}{f_Z(z)}$$

$$f_{S=1|Z}(z) = \frac{p\alpha \cdot e^{-\alpha z}}{p\alpha \cdot e^{-\alpha z} + (1-p)\beta e^{-\beta z}}; \alpha, \beta > 0, z \geq 0$$