

1. $y''' - 3y'' + 4y' + 2y = 2\cosh t = 10e^t + 10e^{-t}$
 Soln will be of the form $y = y_h + y_p$ ($\cosh t = \frac{e^t + e^{-t}}{2}$)
 y_h Homogeneous soln y_p Particular soln.

Homogeneous eqn:-

$$y''' - 3y'' + 4y' + 2y = 0$$

$\Rightarrow$ Characteristic equation: $\lambda^3 - 3\lambda^2 + 4\lambda + 2 = 0$.

Roots are $\lambda_1 = -0.3788 \in \mathbb{R}$.

$$\lambda_2 = a + ib = 1.6894 + 1.5575i$$

$$\lambda_3 = a - ib = \underbrace{1.6894}_a - \underbrace{1.5575i}_b$$

} Complex roots

$$y_h = C_1 e^{\lambda_1 t} + e^{at} (C_2 \cos bt + C_3 \sin bt).$$

(Although roots are complex, we have closed form.
 Thus analytical solution is possible).

Non-homogeneous part:-

Assume particular solution as-

$$y_p = A + B e^t + C e^{-t} + D t e^t + E t e^{-t}$$

(Remember $\cosh t = \frac{e^t + e^{-t}}{2}$)

$$\Rightarrow y_p' = B e^t - C e^{-t} + D e^t + D t e^t + E e^{-t} - E t e^{-t}$$

$$\& y_p'' = B e^t + C e^{-t} + 2D e^t + \cancel{E e^t} + \cancel{D t e^t} - E e^{-t} + E t e^{-t}$$

$$y_p''' = B e^t - C e^{-t} + 3D e^t + D t e^t + E e^{-t} - E t e^{-t}$$

Substitute y_p, y_p', y_p'' & y_p''' into the given differential equation, we get:-

$$\begin{aligned} & Be^t - Ce^{-t} + 3De^t + Dte^t + Ee^{-t} - Ete^{-t} \\ & - 3(Be^t + Ce^{-t} + 2De^t + Dte^t + Ete^{-t}) \\ & + 4(Be^t - Ce^{-t} + De^t + Dte^t + Ee^{-t} - Ete^{-t}) \\ & + 2(A + Be^t + Ce^{-t} + Dte^t + Ete^{-t}) = 10e^t + 10e^{-t}. \end{aligned}$$

$$\begin{aligned} \Rightarrow & 2A + e^t(B + 3D - 3B - 6D + 4B + 4D + 2B) \\ & + e^{-t}(-C + E - 3C - 4C + 4E + 2C) \\ & + te^t(D - 3D + 4D + 2D) \\ & + te^{-t}(-E - 3E - 4E + 2E) = 10e^t + 10e^{-t} \end{aligned}$$

$$\Rightarrow A = 0, D = 0 \text{ \& } E = 0. \text{ (By matching LHS and RHS)}$$

• Coeff. of $e^t = 10$.

$$\Rightarrow 4B = 10 \Rightarrow B = \frac{5}{2}$$

• Coeff of $e^{-t} = 10$.

$$\Rightarrow -6C = 10 \Rightarrow C = -\frac{5}{3}$$

$$\Rightarrow \boxed{y_p = \frac{5}{2}e^t - \frac{5}{3}e^{-t}}$$

Hence general soln for this non-homogeneous diff. equation is -

$$\begin{aligned} y = y_h + y_p &= C_1 e^{at} + e^{at} (C_2 \cos bt + C_3 \sin bt) \\ &+ \frac{5}{2}e^t - \frac{5}{3}e^{-t} \end{aligned}$$