

2. $y''' - 2y'' - y' + 2y = \frac{12e^{2t}}{e^t + 1}$

$$y = y_h + y_p$$

• Homogeneous solⁿ y_h is derived by solving homogeneous eqⁿ.

$$y''' - 2y'' - y' + 2y = 0.$$

$$\Rightarrow \text{Characteristic equation: } - \quad r^3 - 2r^2 - r + 2 = 0$$

$$\Rightarrow (r-1)(r+1)(r-2) = 0.$$

$$\Rightarrow \text{Three roots are } r = -1, 1, 2.$$

Hence y_h is written as -

$$y_h = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$$

• Particular solution is a little difficult to obtain since it's product of e^{2t} & $\frac{1}{e^t + 1}$. $(e^t + 1)$ can only come

in denominator if y_p involves $\log(e^t + 1)$, so that differentiating this gives $\frac{e^t}{e^t + 1}$ form.

Hence y_p is written in the following form -

$$y_p = A + B e^t + C e^t t + D e^{2t} + E e^{2t} \log(1 + e^{-t}) \\ + F e^{2t} \log(1 + e^t) + G e^t \log(1 + e^t) + H e^t \log(1 + e^{-t}) \\ + I e^{-t} \log(1 + e^t) + J e^{-t} \log(1 + e^{-t}) + K t e^{2t}$$

$$y_p' = (B+C) e^t + (t e^t + 2D e^{2t} + 2E e^{2t} \log(1 + e^{-t}) \\ - E e^{2t} \frac{e^{-t}}{(1 + e^{-t})} + 2F e^{2t} \log(1 + e^t) + F e^{2t} \frac{e^t}{1 + e^t} \\ + G e^t \log(1 + e^t) + G e^t \frac{e^t}{1 + e^t} + H e^t \log(1 + e^{-t}) - H e^t \frac{e^{-t}}{1 + e^{-t}} \\ + J e^{-t} \log(1 + e^{-t}) + J e^{-t} \frac{e^{-t}}{1 + e^{-t}} + K e^{2t}$$

$$-Ie^{-t} \log(1+e^t) + Ie^{-t} \frac{e^t}{1+e^t} - Je^{-t} \log(1+e^{-t}) \\ + Je^{-t} \frac{e^{-t}}{1+e^{-t}} + Ke^{2t} + 2kte^{2t}$$

Similarly obtain y_p'' & y_p''' & substitute in the differential equation to find the values of constants a .

$$A = -2$$

B is mixed with C_2 to form a new constant C_2'

$$C = 0$$

D is mixed with C_3 to form a new constant C_3'

$$E = 0$$

$$F = -4$$

$$G = -6$$

$$H = 0$$

$$I = 2$$

$$J = 0$$

$$K = 4$$

Hence $y_p = 4te^{2t} + 2e^{-t} \log(e^t + 1) - 6e^t \log(e^t + 1) \\ - 4e^{2t} \log(e^t + 1) - 2.$

So general solution is -

$$y = y_h + y_p = C_1 e^{-t} + C_2' e^t + C_3' e^{2t} + 4te^{2t} \\ + 2e^{-t} \log(e^t + 1) - 6e^t \log(e^t + 1) \\ - 4e^{2t} \log(e^t + 1) - 2$$
