

Answer:

$$X' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} X$$

Let, $X = X_0 e^{\lambda t}$ be a solution.

X_0 is any arbitrary constant vector.

$$\therefore X' = \lambda X_0 e^{\lambda t} = \lambda X$$

$$\text{so, } \lambda X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} X$$

$$\text{or, } AX = \lambda X \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

$$\Rightarrow (A - \lambda I)X = 0$$

so, this is an eigenvalue problem.

$$\text{so, } \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 2 & -1 & 2-\lambda \end{vmatrix} = 0 \quad \left[\because |A - \lambda I| = 0 \right]$$

$$\Rightarrow (1-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda_1 = 1; \lambda_2 = 1, \lambda_3 = 2$$

$\therefore$ for $\lambda_1 = 1$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{matrix} x_1 = 0; \\ x_2 = x_3 \end{matrix}$$

$$\text{so, } X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{————— (1)}$$

Now, for repeated roots on eigenvalues,

$$\therefore \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} X_2 = X_1$$

$$\text{so, } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow x_1 = 1$$

$$\text{and } 2 - x_2 + x_3 = 1$$

$$\Rightarrow x_2 = x_3 + 1. \quad \text{so, } X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{————— (2)}$$

[$\because$ Let us assume the second solution be $X = e^{\lambda t}(tx_1 + x_2)$ This solution also satisfies the differential equation, where x_2 is a vector]

Now, for $\lambda_3 = 2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 0 ; \text{ ~~} x_2 = 0 \text{ } \text{ and } x_3 = 0 \text{ }~~$$

$$x_1 = x_2 = 0$$

$x_3 = K$ (any arbitrary constant).

$$\text{Let, } x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{--- (3)}$$

So, from (1), (2) and (3), we get the general solution as -

$$x = c_1 x_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} (t x_1 + x_2) + c_3 x_3 e^{\lambda_3 t}$$

$$\text{Or, } x(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 e^{2t} \left(t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

$$\text{Or, } x(t) = e^t \left(c_1 + c_2 t \right) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + e^t c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + e^{2t} c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where c_1, c_2 and c_3 are constants