

11. Solve

$$(e^x \sin y - 2y e^{-x} \sin x) \frac{dx}{dy} + (\cos y + 2e^{-x} \cos x) = 0$$

$$e^x (\sin y - 2y \sin x) dx + (e^x \cos y + 2 \cos x) dy = 0$$

This looks like exact differential equation, so let's check whether it is exact differential equation or not?

For this,

$$\underbrace{(e^x \sin y - 2y \sin x) dx}_M + \underbrace{(e^x \cos y + 2 \cos x) dy}_N = 0$$

Let $M = e^x \sin y - 2y \sin x$

~~$N = e$~~

$N = e^x \cos y + 2 \cos x$

then

$$\frac{\partial M}{\partial y} = e^x \cos y - 2 \sin x$$

and $\frac{\partial N}{\partial x} = e^x \cos y - 2 \sin x$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{exact differential equation})$$

therefore,

$$\frac{\partial f}{\partial x} = e^x \sin y - 2y \sin x \quad \text{--- (a)}$$

$$\frac{\partial f}{\partial y} = e^x \cos y + 2 \cos x$$

Take integration of either $\frac{\partial f}{\partial x}$ or $\frac{\partial f}{\partial y}$, we get.

Take $\frac{\partial f}{\partial y} = e^x \cos y + 2 \cos x$

$$F(x, y) = \int (e^x \cos y + 2 \cos x) dy$$

$$= \cancel{e^x \sin y} + e^x \sin y + 2y \cos x + g(x) \quad \text{--- (b)}$$

Taking derivative of $F(x,y)$ wrt x

$$\frac{\partial F}{\partial x} = \frac{\partial F(x,y)}{\partial x} = e^x \sin y - 2y \sin x + g'(x) \quad \dots (1)$$

Equating this ^{eq (1)} with eq (a)

$$\frac{\partial F}{\partial x} = \frac{\partial F(x,y)}{\partial x}$$

$$\therefore e^x \sin y - 2y \sin x = e^x \sin y - 2y \sin x + g'(x)$$

$$\therefore g'(x) = 0$$

And on integrating $g'(x)$, we get

$$g(x) = 0$$

Putting value of $g(x)$ in eq (b), we get

$$F(x,y) = e^x \sin y + 2y \cos x + 0$$

$\therefore$ final answer is,

$$\boxed{e^x \sin y + 2y \cos x = C}$$

2) Find general solution of

$$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = -8e^{2t} \cos(t) \quad \dots (1)$$

The general solution of above is

$$y(x) = y_p(x) + y_c(x)$$

Note

$$\boxed{\therefore \text{Let } t=x}$$

for $y_c(x)$, we have

$$y^{(4)} - 8y^{(3)} + 26y'' - 40y' + 25y = 0$$

This can be written as

$$a\tau^4 + b\tau^3 + c\tau^2 + d\tau + e = 0$$

then

$$\tau^4 - 8\tau^3 + 26\tau^2 - 40\tau + 25 = 0$$

On solving this,

$$\tau = 2-i, 2+i$$

compare this with $\tau = \alpha + \beta i$, then $\alpha = 2, \beta = 1$

and

$$\therefore y_c(x) = e^x [C_1 \cos x + C_2 \sin x]$$

$$\therefore y_c(x) = e^2 [C_1 \cos x + C_2 \sin x]$$

For $y_p(x)$, we have,

$$y_p(x) = Ae^{2x} [B \cos x + C \sin x]$$

$$= AB e^{2x} \cos x + AC e^{2x} \sin x$$

Let $AB \rightarrow A$ and $AC \rightarrow B$

~~$AB \rightarrow A$~~

~~$AC \rightarrow B$~~

$$\therefore y_p(x) = Ae^{2x} \cos x + Be^{2x} \sin x$$

$$y'_p(x) = A(2 \cos x e^{2x} - e^{2x} \sin x) + B(2 \sin x e^{2x} + e^{2x} \cos x)$$

$$= (2A+B) e^{2x} \cos x + (-A+2B) e^{2x} \sin x$$

$$y''_p(x) = (2A+B)(2 \cos x e^{2x} - e^{2x} \sin x) + (-A+2B)(2 \sin x e^{2x} + e^{2x} \cos x)$$

$$= (3A+4B) e^{2x} \cos x + (-4A+3B) e^{2x} \sin x$$

$$y'''_p(x) = (3A+4B)(2 \cos x e^{2x} - e^{2x} \sin x) + (-4A+3B)(2 \sin x e^{2x} + e^{2x} \cos x)$$

$$= (2A+11B) e^{2x} \cos x + (-11A+2B) e^{2x} \sin x$$

$$y''''_p(x) = (2A+11B)(2 \cos x e^{2x} - e^{2x} \sin x) + (-11A+2B)(2 \sin x e^{2x} + e^{2x} \cos x)$$

$$= (-7A+24B) e^{2x} \cos x + (-24A-7B) e^{2x} \sin x$$

Putting this in eq (9), we get

$$[(-7A+24B) e^{2x} \cos x + (-24A-7B) e^{2x} \sin x] - 8[(2A+11B) e^{2x} \cos x + (-11A+2B) e^{2x} \sin x] + 26[(3A+4B) e^{2x} \cos x + (-4A+3B) e^{2x} \sin x] - 40[(2A+B) e^{2x} \cos x + (-A+2B) e^{2x} \sin x] + 25[A e^{2x} \cos x + B e^{2x} \sin x]$$

$$= -8 e^{2x} \cos x.$$

$$\therefore [(-7A+24B) - 8(2A+11B) + 26(3A+4B) - 40(2A+B) + 25A] e^{2x} \cos x + [(-24A-7B) - 8(-11A+2B) + 26(-4A+3B) - 40(-A+2B) + 25B] e^{2x} \sin x$$

$$= -8 e^{2x} \cos x.$$

$$\therefore (0A+0B) e^{2x} \cos x + (0A+0B) e^{2x} \sin x = -8 e^{2x} \cos x.$$

$$\therefore A=0 \quad B=0$$

$$\therefore y_p(x) = 0$$

Then $y(x) = y_c(x) + y_p(x)$

$$= e^2 (C_1 \cos x + C_2 \sin x) + 0$$

~~$y(x) = e$~~

$\therefore \boxed{y(x) = e^2 (C_1 \cos x + C_2 \sin x)}$

put $x = t$ then

final solution is,

$\boxed{y(t) = e^2 (C_1 \cos t + C_2 \sin t)}$

3). a) consider the linear system

$$x' = -x + 2y$$

$$y' = \alpha x - y$$

a) for what value of α is the equilibrium point $(0,0)$ a stable node? Solution.

An equilibrium point is any point that make all rates 0.

So,

$$x' = -x + 2y = 0$$

$$\therefore x = 2y \quad \text{--- (a)}$$

and

$$y' = \alpha x - y = 0$$

$$\therefore \alpha x = y \quad \text{--- (b)}$$

On solving (a) and (b) we get

$$\alpha x = 2y = y$$

$$\therefore 2\alpha = 1$$

$$\therefore \boxed{\alpha = \frac{1}{2}}$$

(b) for the specified value of $\alpha=2$, determine the general solution by finding the eigenvalues and eigenvectors of the matrix. Sketch the phase portrait.

Solution.

$$x' = -x + 2y$$

$$y' = 2x - y$$

$$\therefore \alpha = 2$$

We know,

$$x' = Ax$$

$$\text{where } A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\text{and } \det(A - \lambda I) = 0$$

where,

$$\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Therefore,

$$\det(A - \lambda I) = \det \begin{bmatrix} -1 - \lambda & 2 \\ 2 & -1 - \lambda \end{bmatrix}$$

$$= (-1 - \lambda)(-1 - \lambda) - 2 \times 2$$

$$= (-1 - \lambda)^2 - 4$$

$$= 1 + 2\lambda + \lambda^2 - 4$$

$$= \lambda^2 + 2\lambda - 3$$

$$= (\lambda + 3)(\lambda - 1)$$

$$\text{and } \det(A - \lambda I) = 0 \rightarrow \text{characteristic equation.}$$

$$\therefore (\lambda + 3)(\lambda - 1) = 0$$

$$\therefore \lambda = -3, 1$$

$$\therefore \text{Eigen values } = \lambda = 1, -3$$

For $\lambda = 1$:

$$(A - \lambda I)k = 0$$

$$\text{or } (A - I)k = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2k_1 + 2k_2 = 0$$

$$2k_1 - 2k_2 = 0$$

or solving this

$$k_1 = 1$$

$$k_2 = 1$$

$$\therefore k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \text{Eigen vector} = k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -3$

$$(A - \lambda I)k = 0$$

$$(A + 3I)k = 0$$

$$\Rightarrow \begin{bmatrix} -1+3 & 2 \\ 2 & -1+3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2k_1 + 2k_2 = 0$$

$$2k_1 + 2k_2 = 0$$

On solving this

$$k_1 = 1$$

$$k_2 = -1$$

$$\therefore \text{Eigen vector} = k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution is

$$x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

For phase portrait, we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{(i)} e^t + \underbrace{c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{(ii)} e^{-3t}$$

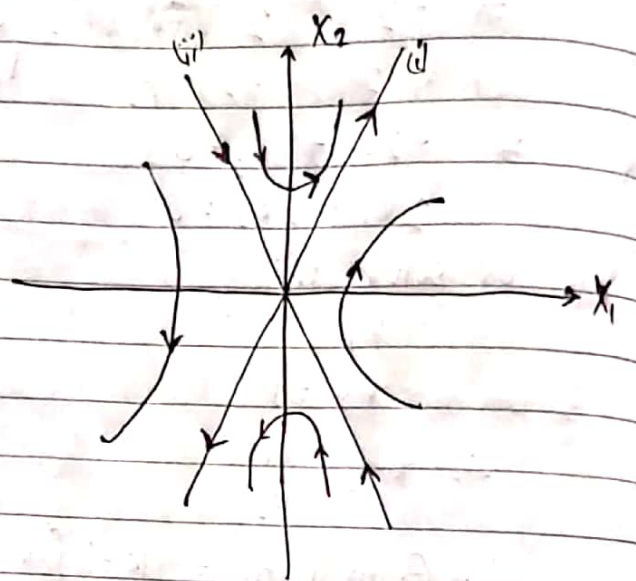
For part (i)

$$x_1 = x_2$$

For part (ii)

$$x_1 = -x_2$$

Sketch phase portrait



This is saddle point.

4). Consider the system of ODE's

$$x' = (x-2)(y-1)$$

$$y' = y(y+x-1)$$

a) Find equilibrium point.

Solution

An equilibrium point is any point that makes all rate 0.

$$x' = (x-2)(y-1) = 0$$

$$\therefore (x-2)(y-1) = 0 \quad \text{--- (a)}$$

and

$$y' = y(y+x-1) = 0$$

$$\therefore y(y+x-1) = 0 \quad \text{--- (b)}$$

From (a),

$$x = 2, \quad y = 1$$

From (b)

$$y = 0, \quad x = 1 - y$$

$$\Rightarrow y = 1 - x$$

When $x=2$ taken from eq (a) and putting in eq (b), we get
 $y = 0, -1$

When $x \cdot y = 1$ taken from eq (a) and putting in eq (b), we get
 $x = 0$

Set of equilibrium points = $\{(0,1), (2,0), (2,-1)\}$.

- (b) Find the linearized system at each of equilibrium points. In each case, compute the eigen values of the corresponding matrix, and sketch the equilibrium points.

Solution.

From part (a),

Equilibrium points are $(0,1)$, $(2,0)$ and $(2,-1)$
 for linearized system and to find eigen value, we have

$$\text{Jacobian matrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

where

$$f(x,y) = x' = (x-2)(y-1) = xy - x - 2y + 2$$

$$g(x,y) = y' = y(y+x-1) = y^2 + xy - y$$

$$\therefore \text{Jacobian matrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} y-1 & x-2 \\ y & 2y+x-1 \end{bmatrix}$$

At point $(0,1)$

$$\text{Jacobian matrix} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = A.$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$\therefore -\lambda(1-\lambda) - (-2)\lambda = 0$$

$$\therefore -\lambda + \lambda^2 + 2 = 0$$

$$\therefore \lambda^2 - \lambda + 2 = 0$$

On solving this, we have

$$\lambda = \frac{1 + \sqrt{-7}}{2}, \frac{1 - \sqrt{-7}}{2}$$

$$\therefore \text{Eigen values } (\lambda) = \frac{1 + \sqrt{-7}}{2}, \frac{1 - \sqrt{-7}}{2}$$

Hence, This is ~~saddle~~ source
for point (2, 0).

$$\text{Jacobian matrix} = A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\therefore \det \begin{bmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$\therefore (-1-\lambda)(1-\lambda) - 0 = 0$$

$$\therefore -1 - \lambda + \lambda + \lambda^2 = 0$$

$$\therefore \lambda^2 - 1 = 0$$

On solving this, we get

$$\lambda = 1, -1$$

$$\therefore \text{Eigen value } |\lambda| = 1, -1$$

Hence, This is saddle

For point (2, -1)

$$\text{Jacobian matrix} = A = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\therefore \det \begin{bmatrix} -2-\lambda & 0 \\ -1 & -1-\lambda \end{bmatrix} = 0$$

$$\therefore (-2-\lambda)(-1-\lambda) - 0 = 0$$

$$\therefore \text{or, } 2 + \lambda + 2\lambda + \lambda^2 = 0$$

$$\text{or } \lambda^2 + 3\lambda + 2 = 0$$

On solving this, we get

$$\lambda = -1, -2$$

$\therefore$ Eigen values $(\lambda) = -1, -2$

Hence, This is sink.

Therefore, $(0, 1) \rightarrow$ ^{source} ~~Saddle~~ $[\because \lambda = \frac{1+\sqrt{-7}}{2}, \frac{1-\sqrt{-7}}{2}]$

$(2, 0) \rightarrow$ saddle $[\because \lambda = 1, -1]$

$(2, -1) \rightarrow$ sink $[\because \lambda = -1, -2]$

- c) Sketch the phase portrait near each critical point.
For point $(0, 1)$

Eigen value $= \lambda = \frac{1+\sqrt{-7}}{2}, \frac{1-\sqrt{-7}}{2}$

then

Eigen vector $(A - \lambda I)k = 0$

where

$$A = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$(A - \lambda I)k = 0$$

$$\Rightarrow \begin{bmatrix} \frac{1+\sqrt{-7}}{2} & -2 \\ 1 & \frac{1-\sqrt{-7}}{2} \end{bmatrix} k = 0 \rightarrow \text{For } \lambda = \frac{1+\sqrt{-7}}{2}$$

$$\therefore k = \begin{bmatrix} x \\ -\frac{1}{4}(1-i\sqrt{7})x \end{bmatrix}$$

Similarly

$$\begin{bmatrix} \frac{1-\sqrt{-7}}{2} & -2 \\ 1 & \frac{1+\sqrt{-7}}{2} \end{bmatrix} k = 0 \rightarrow \text{For } \lambda = \frac{1-\sqrt{-7}}{2}$$

$$\therefore k = \begin{bmatrix} x \\ -\frac{1}{4}(1+i\sqrt{7})x \end{bmatrix}$$

Sketch phase portrait.

