Linear Programming and Optimization

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Aims of the Course

- To introduce the field of optimization by carefully examining the simplest type of problem:
  - maximizing or minimizing a linear function of variables;
  - subject to a collection of linear equality and/or inequality constraints.
- To understand and use the Simplex Method for solving a linear program.
- To develop the dual linear program and use duality to solve the optimization problem.
- To develop and use methods to examine the sensitivity of the solution of a linear program to the data of the problem.

Today: We will see how to formulate a linear program by considering two examples.

Example 1

A farmer who raises chickens would like to determine the amounts of the available ingredients that will meet certain nutritional requirements. The available ingredients and their cost, the nutrients in the ingredients, and the daily requirements are summarized below.

	Ingredient (per lb.)		Minimum Daily
Nutrient	Corn	Alfalfa	Requirement (RDU)
Protein (RDU/lb.)	8	4	10
Carbohydrates (RDU/lb.)	4	4	6
Cost ($/lb.)	0.35	0.20	

What combination of ingredients will meet the nutritional requirements at the least cost?

The Optimization Criterion

As stated in the question, the goal is to *minimize* the cost of the feed mix per chicken.

Defining the Variables: Let

- x denote the number of pounds of Corn and
- y denote the number of pounds of Alfalfa

that the farmer uses to feed a chicken.

Then the function to minimize is the cost

$$\begin{aligned} \zeta(x,y) &= 0.35(\$/lb.)\cdot x(lbs.) + 0.20(\$/lb.)\cdot y(lbs.) \\ &= \$0.35x + \$0.20y. \end{aligned}$$

The Constraints

The chicken must receive the RDU amounts of both protein and carbohydrates.

- *Protein.* The total number of RDUs when x lbs. of corn and y lbs. of alfalfa are used is

$$8(RDU/lb.) \cdot x(lbs.) + 4(RDU/lb.) \cdot y(lbs.) = (8x + 4y)(RDUs).$$

This must be at least the required amount, so it is required that

$$(8x + 4y)(RDUs) \geq 10(RDUs).$$

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- *Carbs*. The RDUs from x lbs. of corn and y lbs. of alfalfa must satisfy

$$(4x + 4y)(RDUs) \geq 6(RDUs).$$

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- *Nonnegativity.* It is obvious that the farmer cannot use a negative number of pounds of either corn or alfalfa. Thus the non-negativity constraints

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \end{aligned}$$

  must also hold.

Concise Statement

The optimization problem can be written simply as follows:

$$\begin{cases} \text{Minimize} & \$0.35x + \$0.20y \\ \text{Subject to} & \\ \text{Protein:} & (8x + 4y)(RDUs) \geq 10(RDUs) \\ \text{Carbs.:} & (4x + 4y)(RDUs) \geq 6(RDUs) \\ \text{Nonneg.:} & x \geq 0, y \geq 0. \end{cases}$$

or if we clean this up by omitting the units in the statement, we get

$$\begin{cases} \text{Minimize} & \zeta(x, y) = 0.35x + 0.20y \\ \text{Subject to} & \\ \text{Protein:} & 8x + 4y \geq 10 \\ \text{Carbs.:} & 4x + 4y \geq 6 \\ \text{Nonneg.:} & x \geq 0, y \geq 0. \end{cases}$$

Example 2

The Sewell Manufacturing Company produces two types of reclining chairs for sale. Two basic types of skilled labor are involved: assembly and finishing. One unit of the top-of-the-line recliner requires $\frac{3}{2}$ hours of assembly, 1 hour of finishing, and sells for a profit of \$275. A unit of the second-line recliner requires $\frac{1}{2}$ hour of assembly, $\frac{1}{2}$ hour of finishing and sells for a profit of \$136. Presently, there are 1000 assembly hours and 800 finishing hours available to the company.

How many of each type of chair should the company manufacture so as to maximize their profit?

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Organizing the Information

	Recliners		
	Top-of-the-Line	Second-Line	Hours Available
Assembly	$\frac{3}{2}$	$\frac{1}{2}$	1000
Finishing	1	$\frac{1}{2}$	800
Profit	\$275	\$136	

Formalizing the Objective

The goal is to determine the number of each type of recliner to make so as to *maximize* the profit earned.

Defining the variables:

- Let x denote the number of Top-of-the-Line recliners to make; and
- let y denote the number of Second-Line recliners to make.

The profit earned from making x and y numbers of these recliners is

$$\zeta(x, y) = \$275x + \$136y.$$

Formalizing the Constraints

The numbers x and y of these recliners must not use any more of the labor than is available for assembly or for finishing.

Assembly Restriction:

$$\frac{3}{2}x + \frac{1}{2}y \leq 1000.$$

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To simplify the inequalities, multiply through by 2 to get

$$\begin{array}{lrcl} \text{Assembly:} & 3x + 1y & \leq & 2000, \\ \text{Finishing:} & 2x + 1y & \leq & 1600. \end{array}$$

The Linear Program

$$\begin{cases} \text{Maximize} & \zeta(x,y) = 275x + 136y \\ \text{Subject to} & \\ \text{assembly:} & 3x + 1y \leq 2000 \\ \text{finishing:} & 2x + 1y \leq 1600 \\ \text{nonneg.:} & x \geq 0, y \geq 0. \end{cases}$$

Notice, we have omitted the units in these expressions. When we include them, the linear program is

$$\begin{cases} \text{Maximize} & \zeta(x,y) = \$(275x + 136y) \\ \text{Subject to} & \\ \text{assembly:} & (3x + 1y) \text{ hours} \leq 2000 \text{ hours} \\ \text{finishing:} & (2x + 1y) \text{ hours} \leq 1600 \text{ hours} \\ \text{nonneg.:} & x \geq 0, y \geq 0. \end{cases}$$

These units are determined on the left-hand side of the inequalities:

- for the Profit by $\left(\frac{\$}{\text{recliner}} \cdot \text{recliners}\right)$; and
- for the Assembly and Finishing by $\left(\frac{\text{hours}}{\text{recliner}} \cdot \text{recliners}\right)$.