an artificial system of symbols whose structure is known. The best-known example of such a symbolism is the so-called system of logistic which was employed by Russell and Whitehead in their *Principia Mathematica*. But it is not necessary that the language in which analysis is carried out should be different from the language analysed. If it were, we should be obliged to suppose, as Russell once suggested, "that every language has a structure concerning which, *in the language*, nothing can be said, but that there may be another language dealing with the structure of the first language, and having itself a new structure, and that to this hierarchy of languages there may be no limit."[1] This was written presumably in the belief that an attempt to refer to the structure of a language in the language itself would lead to the occurrence of logical paradoxes.[2] But Carnap, by actually carrying out such an analysis, has subsequently shown that a language can without self-contradiction be used in the analysis of itself.[3]

CHAPTER IV

THE *A PRIORI*

THE VIEW OF PHILOSOPHY which we have adopted may, I think, fairly be described as a form of empiricism. For it is characteristic of an empiricist to eschew metaphysics, on the ground that every factual proposition must refer to sense-experience. And even if the conception of philosophizing as an activity of analysis is not to be discovered in the traditional theories of empiricists, we have seen that it is implicit in their practice. At the same time, it must be made clear that, in calling ourselves empiricists, we are not avowing a belief in any of the psychological doctrines which are commonly associated with empiricism. For, even if these doctrines were valid, their validity would be independent of the validity of any philosophical thesis. It could

[1] Introduction to L. Wittgenstein's *Tractatus Logico-Philosophicus*, p. 23.

[2] Concerning logical paradoxes, see Russell and Whitehead, *Principia Mathematica*, Introduction, Chapter ii; F. P. Ramsey, *Foundations of Mathematics*, pp. 1–63; and Lewis and Langford, *Symbolic Logic*, Chapter xiii.

[3] Vide *Logische Syntax der Sprache*, Parts I and II.

be established only by observation, and not by the purely logical considerations upon which our empiricism rests.

Having admitted that we are empiricists, we must now deal with the objection that is commonly brought against all forms of empiricism; the objection, namely, that it is impossible on empiricist principles to account for our knowledge of necessary truths. For, as Hume conclusively showed, no general proposition whose validity is subject to the test of actual experience can ever be logically certain. No matter how often it is verified in practice, there still remains the possibility that it will be confuted on some future occasion. The fact that a law has been substantiated in $n-1$ cases affords no logical guarantee that it will be substantiated in the nth case also, no matter how large we take n to be. And this means that no general proposition referring to a matter of fact can ever be shown to be necessarily and universally true. It can at best be a probable hypothesis. And this, we shall find, applies not only to general propositions, but to all propositions which have a factual content. They can none of them ever become logically certain. This conclusion, which we shall elaborate later on, is one which must be accepted by every consistent empiricist. It is often thought to involve him in complete scepticism; but this is not the case. For the fact that the validity of a proposition cannot be logically guaranteed in no way entails that it is irrational for us to believe it. On the contrary, what is irrational is to look for a guarantee where none can be forthcoming; to demand certainty where probability is all that is obtainable. We have already remarked upon this, in referring to the work of Hume. And we shall make the point clearer when we come to treat of probability, in explaining the use which we make of empirical propositions. We shall discover that there is nothing perverse or paradoxical about the view that all the "truths" of science and common sense are hypotheses; and consequently that the fact that it involves this view constitutes no objection to the empiricist thesis.

Where the empiricist does encounter difficulty is in connection with the truths of formal logic and mathematics. For whereas a scientific generalisation is readily admitted to be fallible, the truths of mathematics and logic appear to everyone to be necessary and certain. But if empiricism is correct no proposition which has a factual content can be necessary or certain. Accordingly the empiricist must deal with the truths of logic and mathematics in one of the two following ways: he must say either that they are not necessary truths, in which case he must account for the universal conviction that they are; or he must say that they have no factual content, and then he must explain how a proposition which is empty of all factual content can be true and useful and surprising.

If neither of these courses proves satisfactory, we shall be obliged to give way to rationalism. We shall be obliged to admit that there are some truths about the world which we can know independently of experience; that there are some properties which we can ascribe to all objects, even though we cannot conceivably observe that all objects have them. And we shall have to accept it as a mysterious inexplicable fact that our thought has this power to reveal to us authoritatively the nature of objects which we have never observed. Or else we must accept the Kantian explanation which, apart from the epistemological difficulties which we have already touched on, only pushes the mystery a stage further back.

It is clear that any such concession to rationalism would upset the main argument of this book. For the admission that there were some facts about the world which could be known independently of experience would be incompatible with our fundamental contention that a sentence says nothing unless it is empirically verifiable. And thus the whole force of our attack on metaphysics would be destroyed. It is vital, therefore, for us to be able to show that one or other of the empiricist accounts of the propositions of logic and mathematics is correct. If we are successful in this, we shall have destroyed the foundations of rationalism. For the fundamental tenet of rationalism is that thought is an independent source of knowledge, and is moreover a more trustworthy source of knowledge than experience; indeed some rationalists have gone so far as to say that thought is the only source of knowledge. And the ground for this view is simply that the only necessary truths about the world which are known to us are known through thought and not through experience. So that if we can show either that the truths in question are not necessary or that they are not "truths about the world," we shall be taking away the support on which rationalism rests. We shall be making good the empiricist contention that there are no "truths of reason" which refer to matters of fact.

The course of maintaining that the truths of logic and mathematics are not necessary or certain was adopted by Mill. He maintained that these propositions were inductive generalizations based on an extremely large number of instances. The fact that the number of supporting instances was so very large accounted, in his view, for our believing these generalizations to be necessarily and universally true. The evidence in their favour was so strong that it seemed incredible to us that a contrary instance should ever arise. Nevertheless it was in principle possible for such generalizations to be confuted. They were highly probable, but, being inductive generalizations, they were not certain. The difference between them and the hypotheses of natural science was a difference in degree and not in kind. Experience gave us very good reason to suppose that a "truth" of mathematics or logic was true universally; but we were not possessed of a guarantee. For these "truths" were only empirical hypotheses which had worked particularly well in the past; and, like all empirical hypotheses, they were theoretically fallible.

I do not think that this solution of the empiricist's difficulty with regard to the propositions of logic and mathematics is acceptable. In discussing it, it is necessary to make a distinction which is perhaps already enshrined in Kant's famous dictum that, although there can be no doubt that all our knowledge begins with experience, it does not follow that it all arises out of experience.[1] When we say that the truths of logic are known independently of experience, we are not of course saying that they are innate, in the sense that we are born knowing them. It is obvious that mathematics and logic have to be learned in the same way as chemistry and history have to be learned. Nor are we denying that the first person to discover a given logical or mathematical truth was led to it by an inductive procedure. It is very probable, for example, that the principle of the syllogism was formulated not before but after the validity of syllogistic reasoning had been observed in a number of particular cases. What we are discussing, however, when we say that logical and mathematical truths are known independently of experience, is not a historical question concerning the way in which these truths were originally discovered, nor a psychological question concerning the way in which each of us comes to learn them, but an epistemological

[1] *Critique of Pure Reason*, 2nd ed., Introduction, section i.

question. The contention of Mill's which we reject is that the propositions of logic and mathematics have the same status as empirical hypotheses; that their validity is determined in the same way. We maintain that they are independent of experience in the sense that they do not owe their validity to empirical verification. We may come to discover them through an inductive process; but once we have apprehended them we see that they are necessarily true, that they hold good for every conceivable instance. And this serves to distinguish them from empirical generalizations. For we know that a proposition whose validity depends upon experience cannot be seen to be necessarily and universally true.

In rejecting Mill's theory, we are obliged to be somewhat dogmatic. We can do no more than state the issue clearly and then trust that his contention will be seen to be discrepant with the relevant logical facts. The following considerations may serve to show that of the two ways of dealing with logic and mathematics which are open to the empiricist, the one which Mill adopted is not the one which is correct.

The best way to substantiate our assertion that the truths of formal logic and pure mathematics are necessarily true is to examine cases in which they might seem to be confuted. It might easily happen, for example, that when I came to count what I had taken to be five pairs of objects, I found that they amounted only to nine. And if I wished to mislead people I might say that on this occasion twice five was not ten. But in that case I should not be using the complex sign "$2 \times 5 = 10$" in the way in which it is ordinarily used. I should be taking it not as the expression of a purely mathematical proposition, but as the expression of an empirical generalization, to the effect that whenever I counted what appeared to me to be five pairs of objects I discovered that they were ten in number. This generalization may very well be false. But if it proved false in a given case, one would not say that the mathematical proposition "$2 \times 5 = 10$" had been confuted. One would say that I was wrong in supposing that there were five pairs of objects to start with, or that one of the objects had been taken away while I was counting, or that two of them had coalesced, or that I had counted wrongly. One would adopt as an explanation whatever empirical hypothesis fitted in best with the accredited facts. The one explanation which would in no circumstances be adopted is that ten is not always the product of two and five.

To take another example: if what appears to be a Euclidean triangle is found by measurement not to have angles totalling 180 degrees, we do not say that we have met with an instance which invalidates the mathematical proposition that the sum of the three angles of a Euclidean triangle is 180 degrees. We say that we have measured wrongly, or, more probably, that the triangle we have been measuring is not Euclidean. And this is our procedure in every case in which a mathematical truth might appear to be confuted. We always preserve its validity by adopting some other explanation of the occurrence.

The same thing applies to the principles of formal logic. We may take an example relating to the so-called law of excluded middle, which states that a proposition must be either true or false, or, in other words, that it is impossible that a proposition and its contradictory should neither of them be true. One might suppose that a proposition of the form "*x* has stopped doing *y*" would in certain cases constitute an exception to this law. For instance, if my friend has never yet written to me, it seems fair to say that it is neither true nor false that he has stopped writing to me. But in fact one would refuse to accept such an instance as an invalidation of the law of excluded middle. One would point out that the proposition "My friend has stopped writing to me" is not a simple proposition, but the conjunction of the two propositions "My friend wrote to me in the past" and "My friend does not write to me now": and, furthermore, that the proposition "My friend has not stopped writing to me" is not, as it appears to be, contradictory to "My friend has stopped writing to me," but only contrary to it. For it means "My friend wrote to me in the past, and he still writes to me." When, therefore, we say that such a proposition as "My friend has stopped writing to me" is sometimes neither true nor false, we are speaking inaccurately. For we seem to be saying that neither it nor its contradictory is true. Whereas what we mean, or anyhow should mean, is that neither it nor its apparent contradictory is true. And its apparent contradictory is really only its contrary. Thus we preserve the law of excluded middle by showing that the negating of a sentence does not always yield the contradictory of the proposition originally expressed.

There is no need to give further examples. Whatever instance we care to take, we shall always find that the situations in which a logical or mathematical principle might appear to be confuted are accounted for in such a way as to leave the principle unassailed. And this indicates that Mill was wrong in supposing that a situation could arise which would overthrow a mathematical truth. The principles of logic and mathematics are true universally simply because we never allow them to be anything else. And the reason for this is that we cannot abandon them without contradicting ourselves, without sinning against the rules which govern the use of language, and so making our utterances self-stultifying. In other words, the truths of logic and mathematics are analytic propositions or tautologies. In saying this we are making what will be held to be an extremely controversial statement, and we must now proceed to make its implications clear.

The most familiar definition of an analytic proposition, or judgement, as he called it, is that given by Kant. He said[1] that an analytic judgement was one in which the predicate B belonged to the subject A as something which was covertly contained in the concept of A. He contrasted analytic with synthetic judgements, in which the predicate B lay outside the subject A, although it did stand in connection with it. Analytic judgements, he explains, "add nothing through the predicate to the concept of the subject, but merely break it up into those constituent concepts that have all along been thought in it, although confusedly." Synthetic judgements, on the other hand, "add to the concept of the subject a predicate which has not been in any wise thought in it, and which no analysis could possibly extract from it." Kant gives "all bodies are extended" as an example of an analytic judgement, on the ground that the required predicate can be extracted from the concept of "body," "in accordance with the principle of contradiction"; as an example of a synthetic judgement, he gives "all bodies are heavy." He refers also to "$7+5=12$" as a synthetic judgement, on the ground that the concept of twelve is by no means already thought in merely thinking the union of seven and five. And he appears to regard this as tantamount to saying that the judgement does not rest on the principle of contradiction alone. He holds, also, that through analytic judgements our knowledge is not extended as it is

[1] *Critique of Pure Reason*, 2nd ed., Introduction, sections iv and v.

through synthetic judgements. For in analytic judgements "the concept which I already have is merely set forth and made intelligible to me."

I think that this is a fair summary of Kant's account of the distinction between analytic and synthetic propositions, but I do not think that it succeeds in making the distinction clear. For even if we pass over the difficulties which arise out of the use of the vague term "concept," and the unwarranted assumption that every judgement, as well as every German or English sentence, can be said to have a subject and a predicate, there remains still this crucial defect. Kant does not give one straightforward criterion for distinguishing between analytic and synthetic propositions; he gives two distinct criteria, which are by no means equivalent. Thus his ground for holding that the proposition "$7+5=12$" is synthetic is, as we have seen, that the subjective intension of "$7+5$" does not comprise the subjective intension of "12"; whereas his ground for holding that "all bodies are extended" is an analytic proposition is that it rests on the principle of contradiction alone. That is, he employs a psychological criterion in the first of these examples, and a logical criterion in the second, and takes their equivalence for granted. But, in fact, a proposition which is synthetic according to the former criterion may very well be analytic according to the latter. For, as we have already pointed out, it is possible for symbols to be synonymous without having the same intensional meaning for anyone: and accordingly from the fact that one can think of the sum of seven and five without necessarily thinking of twelve, it by no means follows that the proposition "$7+5=12$" can be denied without self-contradiction. From the rest of his argument, it is clear that it is this logical proposition, and not any psychological proposition, that Kant is really anxious to establish. His use of the psychological criterion leads him to think that he has established it, when he has not.

I think that we can preserve the logical import of Kant's distinction between analytic and synthetic propositions, while avoiding the confusions which mar his actual account of it, if we say that a proposition is analytic when its validity depends solely on the definitions of the symbols it contains, and synthetic when its validity is determined by the facts of experience. Thus, the proposition "There are ants which have established a system of slavery" is a synthetic proposition. For we cannot tell whether it is true or false merely by considering the definitions of the symbols which constitute it. We have to resort to actual observation of the behaviour of ants. On the other hand, the proposition "Either some ants are parasitic or none are" is an analytic proposition. For one need not resort to observation to discover that there either are or are not ants which are parasitic. If one knows what is the function of the words "either," "or," and "not," then one can see that any proposition of the form "Either *p* is true or *p* is not true" is valid, independently of experience. Accordingly, all such propositions are analytic.

It is to be noticed that the proposition "Either some ants are parasitic or none are" provides no information whatsoever about the behaviour of ants, or, indeed, about any matter of fact. And this applies to all analytic propositions. They none of them provide any information about any matter of fact. In other words, they are entirely devoid of factual content. And it is for this reason that no experience can confute them.

When we say that analytic propositions are devoid of factual content, and consequently that they say nothing, we are not suggesting that they are senseless in the way that metaphysical utterances are senseless. For, although they give us no information about any empirical situation, they do enlighten us by illustrating the way in which we use certain symbols. Thus if I say, "Nothing can be coloured in different ways at the same time with respect to the same part of itself," I am not saying anything about the properties of any actual thing; but I am not talking nonsense. I am expressing an analytic proposition, which records our determination to call a colour expanse which differs in quality from a neighbouring colour expanse a different part of a given thing. In other words, I am simply calling attention to the implications of a certain linguistic usage. Similarly, in saying that if all Bretons are Frenchmen, and all Frenchmen Europeans, then all Bretons are Europeans, I am not describing any matter of fact. But I am showing that in the statement that all Bretons are Frenchmen, and all Frenchmen Europeans, the further statement that all Bretons are Europeans is implicitly contained. And I am thereby indicating the convention which governs our usage of the words "if" and "all."

We see, then, that there is a sense in which analytic propositions do give us new knowledge. They call attention to linguistic usages, of which we might otherwise not be conscious, and they reveal unsuspected implications in our assertions and beliefs. But we can see also that there is a sense in which they may be said to add nothing to our knowledge. For they tell us only what we may be said to know already. Thus, if I know that the existence of May Queens is a relic of tree-worship, and I discover that May Queens still exist in England, I can employ the tautology "If p implies q, and p is true, q is true" to show that there still exists a relic of tree-worship in England. But in saying that there are still May Queens in England, and that the existence of May Queens is a relic of tree-worship, I have already asserted the existence in England of a relic of tree-worship. The use of the tautology does, indeed, enable me to make this concealed assertion explicit. But it does not provide me with any new knowledge, in the sense in which empirical evidence that the election of May Queens had been forbidden by law would provide me with new knowledge. If one had to set forth all the information one possessed, with regard to matters of fact, one would not write down any analytic propositions. But one would make use of analytic propositions in compiling one's encyclopædia, and would thus come to include propositions which one would otherwise have overlooked. And, besides enabling one to make one's list of information complete, the formulation of analytic propositions would enable one to make sure that the synthetic propositions of which the list was composed formed a self-consistent system. By showing which ways of combining propositions resulted in contradictions, they would prevent one from including incompatible propositions and so making the list self-stultifying. But in so far as we had actually used such words as "all" and "or" and "not" without falling into self-contradiction, we might be said already to know what was revealed in the formulation of analytic propositions illustrating the rules which govern our usage of these logical particles. So that here again we are justified in saying that analytic propositions do not increase our knowledge.

The analytic character of the truths of formal logic was obscured in the traditional logic through its being insufficiently formalized. For in speaking always of judgements, instead of propositions, and introducing irrelevant psychological questions, the traditional logic gave the impression of being concerned in some specially intimate way with the workings of thought. What it was actually concerned with was the formal relationship of classes, as is shown by the fact that all its principles of inference are subsumed in the Boolean class-calculus, which is subsumed in its turn in the propositional calculus of Russell and Whitehead.[1] Their system, expounded in *Principia Mathematica*, makes it clear that formal logic is not concerned with the properties of men's minds, much less with the properties of material objects, but simply with the possibility of combining propositions by means of logical particles into analytic propositions, and with studying the formal relationship of these analytic propositions, in virtue of which one is deducible from another. Their procedure is to exhibit the propositions of formal logic as a deductive system, based on five primitive propositions, subsequently reduced in number to one. Hereby the distinction between logical truths and principles of inference, which was maintained in the Aristotelian logic, very properly disappears. Every principle of inference is put forward as a logical truth and every logical truth can serve as a principle of inference. The three Aristotelian "laws of thought," the law of identity, the law of excluded middle, and the law of non-contradiction, are incorporated in the system, but they are not considered more important than the other analytic propositions. They are not reckoned among the premises of the system. And the system of Russell and Whitehead itself is probably only one among many possible logics, each of which is composed of tautologies as interesting to the logician as the arbitrarily selected Aristotelian "laws of thought."[2]

A point which is not sufficiently brought out by Russell, if indeed it is recognised by him at all, is that every logical proposition is valid in its own right. Its validity does not depend on its being incorporated in a system, and deduced from certain propositions which are taken as self-evident. The construction of systems of logic is useful as a means of discovering and certifying analytic propositions, but it is not in principle essential even for this purpose. For it is possible to conceive of a symbolism in which every analytic proposition could be seen to be analytic in virtue of its form alone.

The fact that the validity of an analytic proposition in no way

[1] Vide Karl Menger, "Die Neue Logik," *Krise und Neuaufbau in den Exakten Wissenschaften*, pp. 94–6; and Lewis and Langford, *Symbolic Logic*, Chapter v.

[2] Vide Lewis and Langford, *Symbolic Logic*, Chapter vii, for an elaboration of this point.

depends on its being deducible from other analytic propositions is our justification for disregarding the question whether the propositions of mathematics are reducible to propositions of formal logic, in the way that Russell supposed.[1] For even if it is the case that the definition of a cardinal number as a class of classes similar to a given class is circular, and it is not possible to reduce mathematical notions to purely logical notions, it will still remain true that the propositions of mathematics are analytic propositions. They will form a special class of analytic propositions, containing special terms, but they will be none the less analytic for that. For the criterion of an analytic proposition is that its validity should follow simply from the definition of the terms contained in it, and this condition is fulfilled by the propositions of pure mathematics.

The mathematical propositions which one might most pardonably suppose to be synthetic are the propositions of geometry. For it is natural for us to think, as Kant thought, that geometry is the study of the properties of physical space, and consequently that its propositions have factual content. And if we believe this, and also recognise that the truths of geometry are necessary and certain, then we may be inclined to accept Kant's hypothesis that space is the form of intuition of our outer sense, a form imposed by us on the matter of sensation, as the only possible explanation of our *a priori* knowledge of these synthetic propositions. But while the view that pure geometry is concerned with physical space was plausible enough in Kant's day, when the geometry of Euclid was the only geometry known, the subsequent invention of non-Euclidean geometries has shown it to be mistaken. We see now that the axioms of a geometry are simply definitions, and that the theorems of a geometry are simply the logical consequences of these definitions.[2] A geometry is not in itself about physical space; in itself it cannot be said to be "about" anything. But we can use a geometry to reason about physical space. That is to say, once we have given the axioms a physical interpretation, we can proceed to apply the theorems to the objects which satisfy the axioms. Whether a geometry can be applied to the actual physical world or not, is an empirical question which falls outside the scope of the geometry itself. There is no sense, therefore, in asking

[1] Vide *Introduction to Mathematical Philosophy*, Chapter ii.

[2] cf. H. Poincaré, *La Science et l'Hypothèse*, Part II, Chapter iii.

which of the various geometries known to us are false and which are true. In so far as they are all free from contradiction, they are all true. What one can ask is which of them is the most useful on any given occasion, which of them can be applied most easily and most fruitfully to an actual empirical situation. But the proposition which states that a certain application of a geometry is possible is not itself a proposition of that geometry. All that the geometry itself tells us is that if anything can be brought under the definitions, it will also satisfy the theorems. It is therefore a purely logical system, and its propositions are purely analytic propositions.

It might be objected that the use made of diagrams in geometrical treatises shows that geometrical reasoning is not purely abstract and logical, but depends on our intuition of the properties of figures. In fact, however, the use of diagrams is not essential to completely rigorous geometry. The diagrams are introduced as an aid to our reason. They provide us with a particular application of the geometry, and so assist us to perceive the more general truth that the axioms of the geometry involve certain consequences. But the fact that most of us need the help of an example to make us aware of those consequences does not show that the relation between them and the axioms is not a purely logical relation. It shows merely that our intellects are unequal to the task of carrying out very abstract processes of reasoning without the assistance of intuition. In other words, it has no bearing on the nature of geometrical propositions, but is simply an empirical fact about ourselves. Moreover, the appeal to intuition, though generally of psychological value, is also a source of danger to the geometer. He is tempted to make assumptions which are accidentally true of the particular figure he is taking as an illustration, but do not follow from his axioms. It has, indeed, been shown that Euclid himself was guilty of this, and consequently that the presence of the figure is essential to some of his proofs.[1] This shows that his system is not, as he presents it, completely rigorous, although of course it can be made so. It does not show that the presence of the figure is essential to a truly rigorous geometrical proof. To suppose that it did would be to take as a necessary feature of all geometries what is really only an incidental defect in one particular geometrical system.

[1] cf. M. Black, *The Nature of Mathematics*, p. 154.

We conclude, then, that the propositions of pure geometry are analytic. And this leads us to reject Kant's hypothesis that geometry deals with the form of intuition of our outer sense. For the ground for this hypothesis was that it alone explained how the propositions of geometry could be both true *a priori* and synthetic: and we have seen that they are not synthetic. Similarly our view that the propositions of arithmetic are not synthetic but analytic leads us to reject the Kantian hypothesis[1] that arithmetic is concerned with our pure intuition of time, the form of our inner sense. And thus we are able to dismiss Kant's transcendental æsthetic without having to bring forward the epistemological difficulties which it is commonly said to involve. For the only argument which can be brought in favour of Kant's theory is that it alone explains certain "facts." And now we have found that the "facts" which it purports to explain are not facts at all. For while it is true that we have *a priori* knowledge of necessary propositions, it is not true, as Kant supposed, that any of these necessary propositions are synthetic. They are without exception analytic propositions, or, in other words, tautologies.

We have already explained how it is that these analytic propositions are necessary and certain. We saw that the reason why they cannot be confuted in experience is that they do not make any assertion about the empirical world. They simply record our determination to use words in a certain fashion. We cannot deny them without infringing the conventions which are presupposed by our very denial, and so falling into self-contradiction. And this is the sole ground of their necessity. As Wittgenstein puts it, our justification for holding that the world could not conceivably disobey the laws of logic is simply that we could not say of an unlogical world how it would look.[2] And just as the validity of an analytic proposition is independent of the nature of the external world; so is it independent of the nature of our minds. It is perfectly conceivable that we should have employed different linguistic conventions from those which we actually do employ. But whatever these conventions might be, the tautologies in which we recorded them would always be necessary. For any denial of them would be self-stultifying.

[1] This hypothesis is not mentioned in the *Critique of Pure Reason*, but was maintained by Kant at an earlier date.

[2] *Tractatus Logico-Philosophicus*, 3·031.

We see, then, that there is nothing mysterious about the apodeictic certainty of logic and mathematics. Our knowledge that no observation can ever confute the proposition "$7+5=12$" depends simply on the fact that the symbolic expression "$7+5$" is synonymous with "12," just as our knowledge that every oculist is an eye-doctor depends on the fact that the symbol "eye-doctor" is synonymous with "oculist." And the same explanation holds good for every other *a priori* truth.

What is mysterious at first sight is that these tautologies should on occasion be so surprising, that there should be in mathematics and logic the possibility of invention and discovery. As Poincaré says: "If all the assertions which mathematics puts forward can be derived from one another by formal logic, mathematics cannot amount to anything more than an immense tautology. Logical inference can teach us nothing essentially new, and if everything is to proceed from the principle of identity, everything must be reducible to it. But can we really allow that these theorems which fill so many books serve no other purpose than to say in a roundabout fashion 'A = A'?"[1] Poincaré finds this incredible. His own theory is that the sense of invention and discovery in mathematics belongs to it in virtue of mathematical induction, the principle that what is true for the number 1, and true for $n+1$ when it is true for n,[2] is true for all numbers. And he claims that this is a synthetic *a priori* principle. It is, in fact, *a priori*, but it is not synthetic. It is a defining principle of the natural numbers, serving to distinguish them from such numbers as the infinite cardinal numbers, to which it cannot be applied.[3] Moreover, we must remember that discoveries can be made, not only in arithmetic, but also in geometry and formal logic, where no use is made of mathematical induction. So that even if Poincaré were right about mathematical induction, he would not have provided a satisfactory explanation of the paradox that a mere body of tautologies can be so interesting and so surprising.

The true explanation is very simple. The power of logic and mathematics to surprise us depends, like their usefulness, on the limitations of our reason. A being whose intellect was infinitely

[1] *La Science et l'Hypothèse*, Part I, Chapter i.

[2] This was wrongly stated in previous editions as "true for n when it is true for $n+1$."

[3] cf. B. Russell's *Introduction to Mathematical Philosophy*, Chapter iii, p. 27.

powerful would take no interest in logic and mathematics.[1] For he would be able to see at a glance everything that his definitions implied, and, accordingly, could never learn anything from logical inference which he was not fully conscious of already. But our intellects are not of this order. It is only a minute proportion of the consequences of our definitions that we are able to detect at a glance. Even so simple a tautology as "$91 \times 79 = 7189$" is beyond the scope of our immediate apprehension. To assure ourselves that "7189" is synonymous with "91×79" we have to resort to calculation, which is simply a process of tautological transformation—that is, a process by which we change the form of expressions without altering their significance. The multiplication tables are rules for carrying out this process in arithmetic, just as the laws of logic are rules for the tautological transformation of sentences expressed in logical symbolism or in ordinary language. As the process of calculation is carried out more or less mechanically, it is easy for us to make a slip and so unwittingly contradict ourselves. And this accounts for the existence of logical and mathematical "falsehoods," which otherwise might appear paradoxical. Clearly the risk of error in logical reasoning is proportionate to the length and the complexity of the process of calculation. And in the same way, the more complex an analytic proposition is, the more chance it has of interesting and surprising us.

It is easy to see that the danger of error in logical reasoning can be minimized by the introduction of symbolic devices, which enable us to express highly complex tautologies in a conveniently simple form. And this gives us an opportunity for the exercise of invention in the pursuit of logical enquiries. For a well-chosen definition will call our attention to analytic truths, which would otherwise have escaped us. And the framing of definitions which are useful and fruitful may well be regarded as a creative act.

Having thus shown that there is no inexplicable paradox involved in the view that the truths of logic and mathematics are all of them analytic, we may safely adopt it as the only satisfactory explanation of their *a priori* necessity. And in adopting it we vindicate the empiricist claim that there can be no *a priori*

[1] cf. Hans Hahn, "Logik, Mathematik und Naturerkennen," *Einheitswissenschaft*, Heft II, p. 18. "Ein allwissendes Wesen braucht keine Logik und keine Mathematik."

knowledge of reality. For we show that the truths of pure reason, the propositions which we know to be valid independently of all experience, are so only in virtue of their lack of factual content. To say that a proposition is true *a priori* is to say that it is a tautology. And tautologies, though they may serve to guide us in our empirical search for knowledge, do not in themselves contain any information about any matter of fact.

CHAPTER V

TRUTH AND PROBABILITY

HAVING SHOWN how the validity of *a priori* propositions is determined, we shall now put forward the criterion which is used to determine the validity of empirical propositions. In this way we shall complete our theory of truth. For it is easy to see that the purpose of a "theory of truth" is simply to describe the criteria by which the validity of the various kinds of propositions is determined. And as all propositions are either empirical or *a priori*, and we have already dealt with the *a priori*, all that is now required to complete our theory of truth is an indication of the way in which we determine the validity of empirical propositions. And this we shall shortly proceed to give.

But first of all we ought, perhaps, to justify our assumption that the object of a "theory of truth" can only be to show how propositions are validated. For it is commonly supposed that the business of the philosopher who concerns himself with "truth" is to answer the question "What is truth?" and that it is only an answer to this question that can fairly be said to constitute a "theory of truth." But when we come to consider what this famous question actually entails, we find that it is not a question which gives rise to any genuine problem; and consequently that no theory can be required to deal with it.

We have already remarked that all questions of the form, "What is the nature of *x*?" are requests for a definition of a symbol in use, and that to ask for a definition of a symbol *x* in use is to ask how the sentences in which *x* occurs are to be translated into