CHAPTER 1

Introduction to Probability

When and where a bolt of lightning will strike is difficult to predict. How likely do you think it is that you will be struck by lightning at some point in your life? Would this change if you decided to go golfing in a thunderstorm?

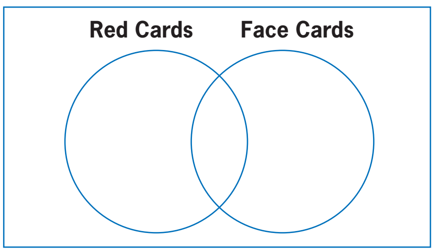
Meteorology is the study of Earth's atmosphere as a way to forecast weather. Although advances in technology have made weather predictions more reliable, there is still a level of uncertainty involved. Meteorologists often use probability, which is the mathematics of uncertain events, in their work. In this chapter, you will learn how to use probability to solve problems involving unknown outcomes in mathematics and in other fields of study.

Key Terms

probability
outcome
experimental probability
subjective probability
theoretical probability
sample space
event
complement
odds in favour
odds against
mutually exclusive events
non-mutually exclusive events
compound events
independent events
dependent events
conditional probability

Literacy Strategy

A Venn diagram is a way to compare the characteristics of a group of objects, which is important when exploring probability. For example, you can use a Venn diagram to compare and sort cards in a standard deck of cards. What do you think the overlap represents?



Career Link

Insurance Underwriter

An insurance underwriter provides automobile insurance to licensed drivers. The terms and cost are determined by a number of factors, such as the driver's age, gender, marital status, and driving record, and the type of vehicle. Who do you think would pay more for insurance, a 30-year-old male with two accidents, or a 20-year-old female with two years of clean driving? Why do you think this is? How are these factors related to probability?

Most insurance underwriters have a bachelor's degree in business, mathematics, or statistics.

Chapter Problem

Game Analysis

Most games usually involve a combination of strategy and chance. The uncertain outcome when you throw a die or spin a spinner adds suspense and excitement. Some games, such as the card game War, are based almost purely on luck. Other games, such as chess, are based almost purely on skill. These games can be very mentally stimulating, but they tend to always favour the more skillful player.

A good game involves an appropriate balance of luck and strategy that matches the entertainment needs of its players. Pick two or three games to research as you study this chapter. At the end of the chapter, you will have the opportunity to share your analysis.

1. What elements of the game involve strategy?
2. What elements of the game involve chance or probability?
3. What is the relative balance of strategy versus chance in this game?

Prerequisite Skills

Fractions, Decimals, and Percent

1. Express each fraction as a decimal and as a percent.

a) $\frac{1}{4}$

b) $\frac{5}{6}$

c) $\frac{2}{3}$

d) $\frac{13}{20}$

2. Express each fraction in lowest terms.

a) $\frac{9}{12}$

b) $\frac{13}{52}$

c) $\frac{22}{35}$

d) $\frac{16}{36}$

3. Add or subtract the following. Express your answer as a fraction in lowest terms, as a decimal, and as a percent.

> **Example:**
>
> $\frac{2}{3}+\frac{1}{4}=\frac{8}{12}+\frac{3}{12}$
>
> $=\frac{11}{12}$
>
> Express fractions with a common denominator. Add numerators. Is the fraction in lowest terms?
>
> To express a fraction as a decimal, divide the numerator by the denominator.
>
> $11 \div 12 = 0.91\overline{6}$
>
> To express a decimal as a percent, multiply by 100%.
>
> $0.91\overline{6} \times 100\% = 91.\overline{6}\%$, or approximately 92%

a) $\frac{1}{6}+\frac{1}{3}$

b) $\frac{1}{4}+\frac{4}{6}$

c) $\frac{3}{4}-\frac{1}{3}$

d) $1-\frac{1}{4}$

4. Multiply. Express your answer as a fraction in lowest terms, as a decimal, and as a percent.

> **Example:**
>
> $\frac{3}{4}\times\frac{1}{6}=\frac{3}{24}$
>
> $=\frac{1}{8}$
>
> Multiply numerators and denominators. Is the fraction in lowest terms?
>
> To express as a decimal, divide the numerator by the denominator.
>
> $1 \div 8 = 0.125$
>
> To express as a percent, multiply by 100%.
>
> $0.125 \times 100\% = 12.5\%$

a) $\frac{1}{6}\times\frac{1}{2}$

b) $\frac{1}{4}\times\frac{2}{3}$

c) $\frac{2}{3}\times\frac{5}{6}$

d) $\frac{5}{12}\times\frac{3}{10}$

Ratio and Proportion

5. A bag contains 3 red counters, 2 blue counters, and 5 yellow counters.

a) Write a ratio that expresses the number of red counters to the total number of counters.

b) Repeat a) for the other two colours. Write each ratio in lowest terms.

c) What percent of the total number of counters does each colour represent?

6. A baseball player has 10 hits in 35 times at bat.

a) Express the ratio of hits to times at bat in fraction form.

b) Convert the fraction to a decimal, rounded to three decimal places.

c) Use proportional reasoning to estimate the number of hits this player would have in 400 times at bat.

Randomization

A random act is an occurrence in which the outcome is unpredictable.

7. Classify each act as either random or non-random. Explain your reasoning.

a) Flipping a coin

b) Safely entering a traffic intersection

c) Looking into a box and picking your favourite candy

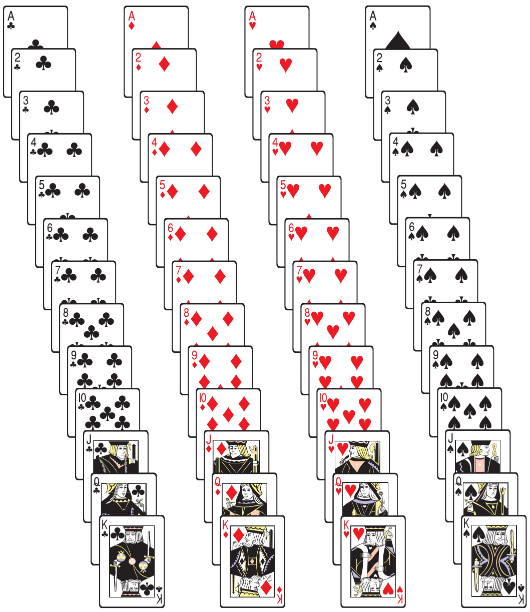
d) Reaching into a box and picking a candy without looking

8. **a)** Describe a random act scenario in a board game.

b) Describe a scenario that involves a non-random act.

Playing Cards and Dice

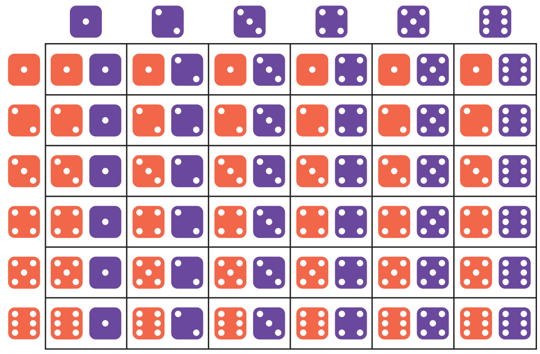
9. A standard deck of playing cards has four suits: clubs, diamonds, hearts, and spades.



a) What fraction of the deck are spades?

b) Face cards are any cards showing a face, namely a jack, queen, or king. What percent of the deck are red face cards?

10. When you throw a pair of standard dice, the value shown on the upper faces gives the outcome of that throw. The following table illustrates all possible outcomes.



List and count all the ways each of the following sums could occur.

a) 2 **b)** 7 **c)** 1

d) doubles (both dice produce the same value)

e) a perfect square

Organizing, Presenting, and Analysing Data

A class was surveyed to determine the students' favourite cafeteria menu items. The tally sheet shows the results.

Menu Item	Favourite
Lasagna	卌 |
Meatloaf	卌
Fish	|||
Pork chops	|||
Chicken	卌 卌

Use this information to answer #11 and #12.

11. Construct a bar graph to represent the data.

12. **a)** Which meal is the class's favourite? What percent of the class chose this meal?

b) What fraction of the class did not choose lasagna?

c) **Open Question** Ask and answer a question related to the data.

1.1

Simple Probabilities

Learning Goal

I am learning to

- use probability to describe the likelihood of something occurring
- measure and calculate simple probabilities

Minds On...

The students know there are 10 coloured counters in the bag; however, they do not know how many of each colour there are.

- How could they estimate the number of each colour?
- What mathematical processes could they use to determine the bag's contents?

Action!

Probability involves making predictions about uncertain situations (or events). A probability experiment is a specific action that has at least two possible results, or **outcomes**. The probability of a certain outcome represents how likely it is to occur.

One way to measure probability is to carry out a probability experiment several times and analyse the results using statistics. Statistics involves working with large amounts of data. Probability and statistics are closely related and are often studied together. The **experimental probability** of an outcome is a measure of how frequently it occurs in a probability experiment. You can express probability as a fraction, decimal, or percent.

Experimental Probability

$$P(A) = \frac{n(A)}{n(T)}$$

where $P(A)$ is the probability that outcome A occurs, $n(A)$ is the number of times that outcome A occurred, and $n(T)$ is the total number of trials.

probability

- likelihood of something occurring

outcome

- a possible result of an experiment

experimental probability

- probability based on experimental trials
- number of times an outcome happens divided by total number of trials
- sometimes called *statistical probability* or *empirical probability*

Investigate Experimental Probability

1. Work with a partner. Secretly create a "mystery bag" of 10 counters using a combination of three different colours of your choice. Your partner does the same.

2. Take turns drawing a single counter at random from your partner's mystery bag. Replace the counter after each draw and shake the bag. No peeking! Record your results in a table, like the one shown.

Colour	Outcomes
Blue (B)	|||
Red (R)	|
Yellow (Y)	~~||||~~ |

3. **a)** Carry out 10 trials.

   b) Determine the experimental probability of drawing each colour listed in your table. Use these values to make a prediction about the contents of your partner's mystery bag.

   c) How confident are you with your prediction? Explain.

4. **a)** Reveal the contents and check your prediction. How close was it?

   b) Compare results with your partner and with other classmates. Did everyone have the same results? Explain.

5. **Reflect**

   a) Explain why using experimental probability is not always an accurate method of predicting.

   b) Can you think of some ways to improve the accuracy of experimental probability? Explain.

6. **Extend Your Understanding** How accurate would the experimental probability values be if none of the 10 counters drawn were replaced before drawing the next counters? Explain.

Materials

- coloured counters (e.g., tiles, cubes, etc.)
- paper bag or envelope

Literacy Link

At random means that each counter has an equal likelihood of being chosen.

Processes

Selecting Tools and Computational Strategies

To calculate the experimental probability, use $P(A) = \frac{n(A)}{n(T)}$.

For example, from the table,

$$P(B) = \frac{n(B)}{n(T)} = \frac{3}{10}$$

So, the probability of selecting a blue counter is $\frac{3}{10}$.

Example 1

Calculate Experimental Probability

A student spins a mystery spinner 24 times. The table shows the results.

Colour	Favourable Outcomes, $n(A)$
Red	12
Yellow	4
Blue	8

a) Determine the experimental probability of the spinner landing on each colour. Express your answers as a fraction, a decimal, and a percent.

b) Determine the sum of the probabilities and explain what it means.

c) What could this spinner look like? Can you be certain this is what the spinner looks like?

Solution

a) To calculate the experimental probability, divide the number of favourable outcomes by the total number of trials.

The total number of trials is:

$$\begin{aligned} n(T) &= 12 + 4 + 8 \\ &= 24 \end{aligned}$$

The experimental probability of the spinner landing on red is

$$\begin{aligned} P(R) &= \frac{n(R)}{n(T)} \\ &= \frac{12}{24} \\ &= \frac{1}{2} \end{aligned}$$

The experimental probability of the spinner landing on red is $\frac{1}{2}$, or 0.5, or 50%.

Similarly,

$$\begin{aligned} P(Y) &= \frac{n(Y)}{n(T)} \\ &= \frac{4}{24} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} P(B) &= \frac{n(B)}{n(T)} \\ &= \frac{8}{24} \\ &= \frac{1}{3} \end{aligned}$$

The experimental probability of the spinner landing on yellow is $\frac{1}{6}$, or $0.1\overline{6}$, or approximately 17%. The experimental probability of the spinner landing on blue is $\frac{1}{3}$, or $0.\overline{3}$, or approximately 33%.

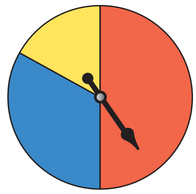
b) Add all the probability values. Use the unreduced forms since they already have a common denominator.

$$\begin{aligned} P(R) + P(Y) + P(B) &= \frac{12}{24} + \frac{4}{24} + \frac{8}{24} \\ &= \frac{24}{24} \\ &= 1 \end{aligned}$$

Why does this result make sense?

c) Based on the statistical probabilities, the spinner could be one-half red, one-sixth yellow, and one-third blue.

This spinner design may or may not look like the real spinner because it is based on experimental probability. Later in this chapter you will learn how to improve the accuracy of experimental probability.



Processes

Reflecting and Representing
How is a spinner related to a pie graph? What does a spinner or pie graph show more clearly than a table does?

Your Turn

A mystery spinner produces these results.

Colour	Favourable Outcomes, $n(A)$
Orange	8
Red	4
Purple	8
Green	12

a) Determine the experimental probability of the spinner landing on each colour. Express your answers as a fraction, a decimal, and a percent.

b) What could this spinner look like?

c) Is it possible that there is a fifth colour? Explain your answer.

If you add all probabilities in an experiment, the total will always equal 1. Each probability represents the fraction of times an outcome occurred. All the fractions combined make up the whole set of outcomes that occurred during the experiment.

Sum of Probabilities

For a probability experiment in which there are n outcomes,

$P_1 + P_2 + P_3 + \dots P_n = 1$

where $P_1, P_2, P_3, \dots P_n$ are the probabilities of the individual outcomes.

Experimental probability is a useful tool for making predictions. Although the predictions may not always be perfectly correct, they can be close enough to help with decision making.

Example 2

Apply Experimental Probability

Tia is the manager of a pizza shop. The table shows the number of pizza slices ordered during the lunch rush over several days.

Pizza Type	Number of Slices
Pepperoni	98
Hawaiian	48
Vegetarian	51

a) Determine the experimental probability that a customer will order each type of pizza slice.

b) Based on this information, what advice should Tia offer her chef in order to be ready for the lunch rush?

Solution

a) Determine the total number of slices ordered.

$$n(T) = 98 + 48 + 51$$
$$= 197$$

Use this value to determine the experimental probability for each type of pizza.

Pizza Type	Number of Slices	Experimental Probability, $\frac{n(A)}{n(T)}$
Pepperoni	98	$\frac{98}{197} = 0.497...$
Hawaiian	48	$\frac{48}{197} = 0.243...$
Vegetarian	51	$\frac{51}{197} = 0.258...$

The probability of a customer choosing a pepperoni slice is about 50%, a Hawaiian slice is about 24%, and a vegetarian slice is about 26%.

b) These results show that the shop sells about the same number of Hawaiian and vegetarian pizza slices and about twice as many pepperoni slices. Tia should advise her chef to bake two pepperoni pizzas for every Hawaiian pizza and vegetarian pizza in order to be ready for the lunch rush.

Your Turn

A market researcher is conducting a telephone poll to gather data about which type of television service families use the most. The table illustrates the results.

Television Service	Tally
Cable	48
Satellite	42
Internet	15
Antenna	4
None	6

a) Determine the experimental probability of using each television service the most.

b) Who might be interested in these results, and for what purpose?

c) Suggest how these results may change over time. Explain why you think so.

Probability is often used outside of mathematics, both formally and informally. Suppose your friend tells you, "I'm 99% certain I just aced my history test!" It is quite likely that she did. This is an example of **subjective probability**. In this case, your friend most likely felt that she knew the answers to all or most of the questions on the test.

subjective probability

- a probability estimate based on intuition
- often involves little or no mathematical data

Subjective probability has no mathematical definition. This use of probability is related to relative certainty. The probability of an outcome can range anywhere between 0 (impossible) and 1 (certain).

Example 3

Estimate Subjective Probability

Match each scenario with its most likely subjective probability.

	Scenario	Subjective Probability, *P*(*A*)
a)	A person randomly selected from your high school is a student.	
b)	A shaker randomly picked from a dining room table contains pepper.	0.2 0.9 0.5
c)	You turn on the radio at some random time and an advertisement is playing.	

Solution

Consider the likelihood of each scenario.

a) Most high schools have far more students than teachers and other staff. Therefore, the probability of randomly selecting a student will be close to 1. A good subjective probability of this event occurring is 0.9.

b) Most dining room tables have the same number of salt and pepper shakers. Therefore, a reasonable subjective probability of randomly picking a pepper shaker is 0.5.

c) Advertisements are played often on most radio stations, but not nearly as much as music. Therefore, a subjective probability of randomly encountering a radio advertisement could be around 0.2.

Your Turn

Estimate the subjective probability of each of the following outcomes. Justify your estimates.

a) You will have a snow day in July where you live.

b) The sun will set in the west tonight.

c) The next person to enter the school cafeteria will be female.

Consolidate and Debrief

Key Concepts

- The probability of an outcome is a measure of how likely it is to occur in a probability experiment.
- The experimental probability of an outcome is based on experimental data. It is defined as the number of favourable outcomes divided by the total number of trials.

$$P(A) = \frac{n(A)}{n(T)}$$

- Subjective probability is an estimate of how likely it is something will occur, based largely on intuition.

Reflect

R1. a) What is meant by the term "experimental probability"? Explain how it is calculated.

b) Explain why experimental probability is a useful strategy for making predictions.

c) Explain why experimental probability is not a perfect strategy for making accurate predictions.

R2. a) The probability of an outcome is 0. What does this mean?

b) The probability of an outcome is 1. What does this mean?

c) Why does the probability of an outcome always have a value between 0 and 1?

R3. a) Write a tweet that explains the concept of subjective probability.

b) Write another tweet that provides an example of subjective probability.

Literacy Link

A *tweet* is a message that is no longer than 140 keystroke characters. Hashtags are sometimes used to identify and flag important words, such as #probability.

Practise

Choose the best answer for #1.

1. A standard die is rolled 12 times and a 2 comes up 3 times. The experimental probability of rolling a 2 with this die is

A 0.17

B 0.20

C 0.25

D 0.33

2. A coin is tossed 10 times and comes up heads 4 times. What is the experimental probability of this coin coming up

a) heads?

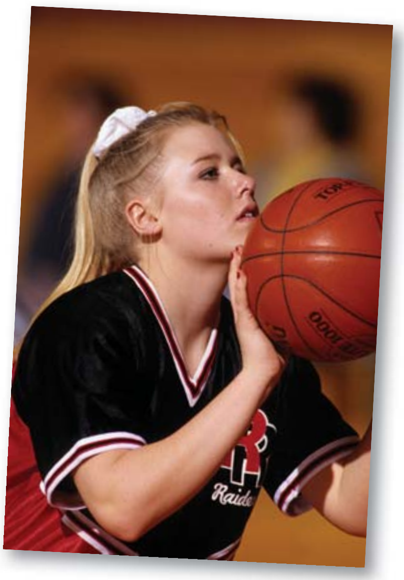
b) tails?

Apply

3. a) Helena successfully made 21 out of 30 free throw attempts. What is the experimental probability that she can make a successful free throw?

b) If Helena makes 5 out of the next 10 shots, what is her new overall experimental probability of scoring?

c) Helena says that she typically makes 80% of her free throws. How accurate would you say this statement is? Justify your answer.



4. Communication The table shows the results for a mystery spinner.

Colour	Favourable Outcomes, $n(A)$
Yellow	6
Green	2
Purple	3
Blue	1

a) Determine the experimental probability of the spinner landing on each colour.

b) Draw what the spinner could look like, based on the given data.

c) Could the real spinner look different? Explain.

5. Open Question

a) Create the results for a mystery spinner where the experimental probability of the spinner landing on green is twice as great as landing on orange.

b) Draw what the spinner could look like, based on your data.

6. An ice-cream stand owner keeps track of his cone sales over a period of several days. The table shows the results.

Flavour	Number of Sales
Vanilla	9
Chocolate	21
Raspberry ripple	43
Pralines and cream	78

a) Determine the experimental probability of a random customer ordering each flavour.

b) How could this information be useful for the ice-cream stand owner?

7. **Application** A weather report claims that the PoP of a rainy day in the previous April was 70%. How many rainy days were there in April?

Literacy Link

In meteorology, the study of weather, *PoP* means the probability of precipitation. It represents the probability that precipitation (rain, snow, etc.) will occur.

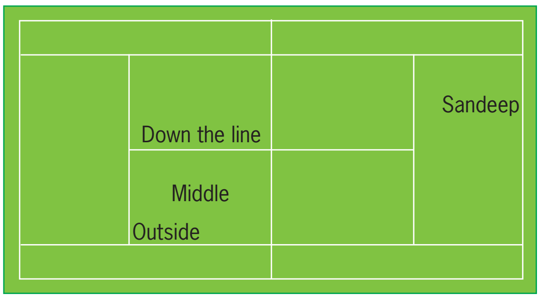
8. **Application** A pitcher throws the following pitches in a game.

Pitch	Count
Fast ball	86
Curve ball	8
Knuckle ball	0

a) Determine the experimental probability that the pitcher will throw each type of pitch.

b) How could this information be useful to the batter?

9. When Sandeep makes her first serve in tennis, she aims for one of the three regions indicated below.



The table shows the ball placements for several of her previous first serves, not counting missed serves.

Serve	Count
Down the line	3
Middle	12
Outside	25

a) Determine the experimental probability of each of Sandeep's serve locations.

b) How could this information be useful to Sandeep's opponent?

10. **Communication** Estimate the subjective probability of each event. Justify your reasoning.

a) You will earn an A in this course.

b) You will pass this course.

c) It will snow tomorrow.

d) You will hear your favourite song on the radio within the next week.

11. **Open Question** Pick a favourite sports team or athlete. Identify an upcoming competition (e.g., the Stanley Cup playoffs or the next Olympics). Estimate the subjective probability that your team or athlete will win their competition. Justify your estimate.

12. **Thinking** The experimental probability of randomly choosing a male from a litter of puppies is $\frac{1}{4}$. If there are 6 female puppies in the litter, how many males are there?

✓ Achievement Check

13. A paper bag contains 10 coloured counters. A counter is randomly drawn and then replaced for several trials. The table shows the results.

Colour	Frequency
Red	22
Green	75
Orange	64
Blue	39

a) Construct a bar graph or pie chart of the data, with or without technology.

b) Determine the experimental probability of randomly drawing each colour.

c) Use these results to predict the contents of the bag. Explain your reasoning.

d) Is it possible that your answer to c) could be incorrect? Explain.

14. **Thinking** A spinner has four colours: red, yellow, green, and purple. The experimental probability of spinning a red is 0.2, of spinning a yellow is 0.3, and of spinning a green is 0.4. If the spinner was spun 30 times, how many times did it land on purple?

15. You can use a graphing calculator to generate random numbers to simulate a probability experiment. Suppose you wish to simulate rolling an 8-sided die 20 times. Follow these instructions to carry out this experiment:

- Press MATH. From the PRB menu, choose **5: randInt(**.
- Enter the following:

```
randInt(1,8,20)
```

The calculator will randomly produce 20 integer values between the values of 1 and 8.

a) Carry out the experiment described above by pressing ENTER.

b) Tally the outcomes. Use the left ⟵ and right ⟶ arrow keys to scroll through the trials.

c) Determine the experimental probability of rolling each value from 1 to 8.

d) Are these values all equal? Do you think they should be? Why or why not?

16. Use the random number generator of a graphing calculator to determine the experimental probability of rolling a 2 with a 4-sided die, using 10 trials.

Extend

17. **Open Question** Design a probability experiment that will reveal purchasing trends at your school cafeteria. Carry out the experiment and calculate experimental probabilities related to your study. You may wish to focus your study on one particular area, such as drink purchases or main menu selections. Write a brief report of your findings, including how the information could be useful.

18. A grocery store manager recorded these data at various times during a typical weekday.

Time of Day	Customers per Hour	Cash Registers Open
Morning	99	5
Afternoon	204	8
Evening	58	4
Overnight	16	1

On average, it takes 3 min to check out a customer. Use experimental probability to determine whether these staffing levels are appropriate or whether some changes should be made. Justify your reasoning.

1.2

Theoretical Probability

Learning Goal

I am learning to

- calculate theoretical probability

Minds On...

Board games usually involve a combination of strategy and luck. Some board games use a pair of standard dice.

- How many possible outcomes are there when rolling a pair of standard dice?
- Suppose you are playing a game involving the sum of two dice. Do you think all sums are equally likely?
- Are there different outcomes that can produce a sum of 2?
- What about a sum of 7?

Action!

The **theoretical probability** of an outcome is one based on analysing all possible outcomes. Unlike experimental probability, no experiment is carried out. All possible outcomes combined make up the **sample space**. It is often useful to combine different outcomes that have something in common. An **event** occurs when any of these similar outcomes occur. For example, the dice pictured above show a sum of 7. But this is not the only outcome that can result in a sum of 7. What are some others?

If all outcomes are equally likely, then the theoretical probability of an event, A, is a measure of the ratio of the number of ways it can occur compared to the entire sample space. You can express this probability as a fraction, decimal, or percent.

Theoretical Probability

$$P(A) = \frac{n(A)}{n(S)}$$

where $P(A)$ is the probability that event A can occur, $n(A)$ is the number of ways it can occur, and $n(S)$ is the total number of possible outcomes in the sample space.

theoretical probability
- probability based on analysis of all possible outcomes
- also called *classical probability*

sample space
- collection of all possible outcomes
- sometimes called *sample set*

event
- set of outcomes that have a common characteristic

Investigate Outcomes and Events

Materials

- 2 standard dice

1. This table illustrates the possible outcomes when two standard dice are thrown.

	Outcome	Die 1					
		⚀	⚁	⚂	⚃	⚄	⚅
Die 2	⚀	⚀ ⚀	⚀ ⚁	⚀ ⚂	⚀ ⚃	⚀ ⚄	⚀ ⚅
	⚁	⚁ ⚀	⚁ ⚁	⚁ ⚂	⚁ ⚃	⚁ ⚄	⚁ ⚅
	⚂	⚂ ⚀	⚂ ⚁	⚂ ⚂	⚂ ⚃	⚂ ⚄	⚂ ⚅
	⚃	⚃ ⚀	⚃ ⚁	⚃ ⚂	⚃ ⚃	⚃ ⚄	⚃ ⚅
	⚄	⚄ ⚀	⚄ ⚁	⚄ ⚂	⚄ ⚃	⚄ ⚄	⚄ ⚅
	⚅	⚅ ⚀	⚅ ⚁	⚅ ⚂	⚅ ⚃	⚅ ⚄	⚅ ⚅

a) Which sum or sums has the greatest theoretical probability? What is the value of this probability?

b) Which sum or sums has the lowest probability? What is the value of this probability?

2. What is the probability of rolling a 9 or greater?

3. **Open Question**

a) Create your own probability problem based on the data in the table.

b) Trade problems with a classmate and solve each other's problems. Compare your solutions.

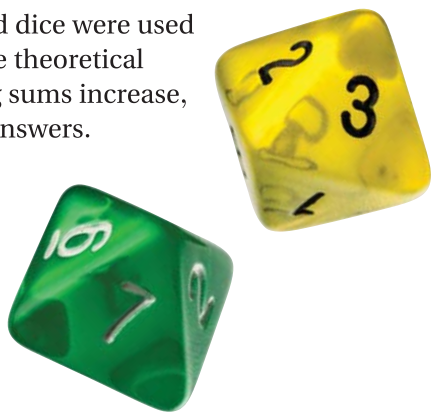
4. **Reflect** Explain how the table in this investigation was useful for calculating theoretical probabilities.

5. **Extend Your Understanding** Suppose 8-sided dice were used instead, numbered 1 through 8. Would the theoretical probability of rolling each of the following sums increase, decrease, or stay the same? Explain your answers.

a) 2

b) 9

c) doubles



In the investigation, you used a table to represent the sample space of the probability experiment. You can also use tree diagrams and Venn diagrams to help organize the outcomes of a sample space. The choice of strategy to use often depends on the situation.

Literacy Link

When using *set notation* to describe a list, write the elements in a logical order, separated by commas. Use set parentheses.

For example, $S = \{1, 2, 3, 4, 5, 6\}$.

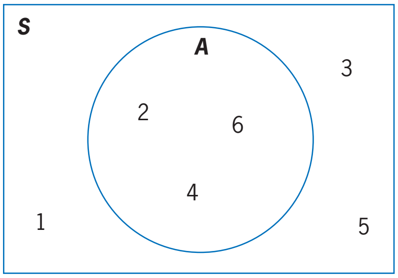
You can also represent a sample space and event of a probability experiment using set notation. Suppose, for example, that a number cube is rolled and the favourable event, A, is rolling an even number. The sample space is all six possible outcomes:

$S = \{1, 2, 3, 4, 5, 6\}$

The event of rolling an even number includes these three outcomes:

$A = \{2, 4, 6\}$

You can use a Venn diagram to represent this relationship visually.



Example 1

Calculate Theoretical Probability

Fiona has two shirts and three pairs of pants that she can wear to her co-op placement.

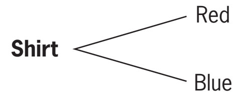
a) Suppose Fiona randomly picks a shirt and a pair of pants. Identify the sample space for this probability experiment.

b) Fiona does not like to wear blue and green together, nor does she like orange and red together. What is the theoretical probability that she will randomly pick a shirt and pants combination that she does not like?

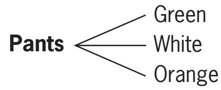
c) Suppose Fiona wants to decrease the probability of drawing a combination she does not like. If Fiona buys another pair of pants, which colour should she choose—green, white, or orange—and why?

Solution

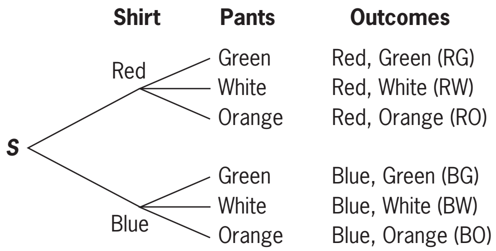
a) Construct a tree diagram to organize all possible outcomes. Then analyse the results. Fiona has two shirts, which you can represent using two branches.



Fiona also has three pairs of pants, which you can represent using three branches.



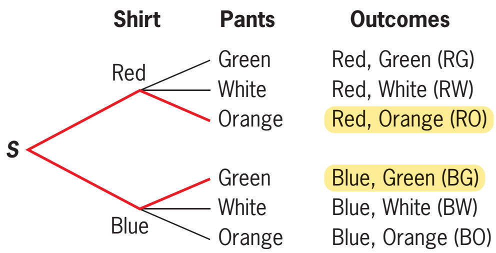
Combine these to illustrate all possible outcomes.



Read the sample space by following each branch in turn.

$S = \{\text{RG, RW, RO, BG, BW, BO}\}$

b) The sample space is all six possible outcomes. The event of drawing a combination Fiona does not like, C, consists of two outcomes: the red-orange combination and the blue-green combination.



The tree diagram shows two out of six possible outcomes resulting in a combination Fiona does not like.

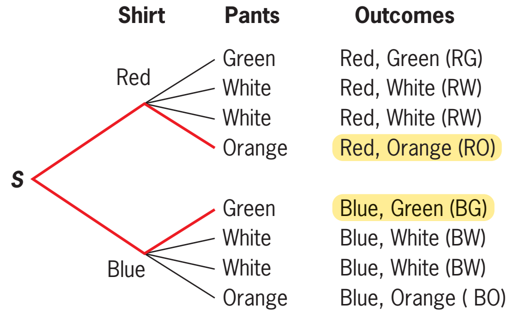
$C = \{\text{BG, RO}\}$

The probability of having a combination Fiona does not like, $P(C)$, is

$$P(C) = \frac{n(C)}{n(S)}$$
$$= \frac{2}{6}$$
$$= \frac{1}{3}$$

Therefore, there is a $\frac{1}{3}$, or approximately 33%, theoretical probability that Fiona will randomly draw a combination she does not like.

c) If Fiona buys a fourth pair of pants from among the same colours that she already has, she should buy a pair that she can wear with all her shirts. Green and orange both produce a combination she does not like, but white does not. The tree diagram shows the sample space if she buys a second pair of white pants.



In this scenario, a combination she does not like can occur in two possible ways out of eight. Therefore, the probability of Fiona randomly drawing an outfit she does not like decreases from 33% to 25%.

Your Turn

Lee has a green hat, a black hat, and a grey hat. She also has green gloves, black gloves, and red gloves. In a hurry, Lee randomly grabs a hat and gloves. Determine the theoretical probability that the hat and gloves are the same colour.

Sometimes you need to know the probability that one event happens compared to all others. If one event is A, then the event A' is all of the possible outcomes not in A. This is known as the complement of A. Because the sum of all probabilities in a sample space must equal 1, there is a useful relationship between $P(A)$ and $P(A')$.

complement
- set of possible outcomes not included in an event

Literacy Link

The event A' is called *"A prime."*

$P(A) + P(A') = 1$

This relationship can be rearranged into two other useful forms.

$P(A') = 1 - P(A)$ or $P(A) = 1 - P(A')$

Example 2

Probability of a Complement

Battleship is a game in which two opponents use a coordinate grid to try to guess the location of each other's ships.

Examine the board shown. Suppose a location is guessed at random.

a) What is the theoretical probability of hitting a ship on the first guess?

b) What is the probability of a miss on the first guess?

Solution

a) Let A represent the event of hitting a ship.

Determine the number of locations covered by ships, $n(A)$.

$$n(A) = 2 + 3 + 3 + 4 + 5$$
$$= 17$$

The grid has dimensions 10 by 10. So, there are $n(S) = 100$ locations in total.

The probability of hitting a ship is

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{17}{100}$$

The theoretical probability of randomly hitting a ship on the first guess is 17%, or 0.17.

b) A random guess will result in either a hit, A, or a miss, A'. These are complementary events. One way to calculate the probability of missing the ships on the first guess is to count all locations not covered by a ship and divide by 100. A more efficient strategy, however, is to apply complementary reasoning.

$$P(A') = 1 - P(A)$$
$$= 1 - 0.17$$
$$= 0.83$$

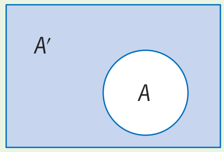
Therefore, there is an 83% chance of randomly missing the ships.

Your Turn

A box contains 3 bran, 4 banana, 5 blueberry, and 3 carrot muffins. What is the theoretical probability that you will not randomly choose a blueberry muffin?

Processes

Representing
You can visually represent the relationship between A and A', as in the Venn diagram shown. The two regions combined make up the sample space representing all outcomes.



One application of probability, often used in sports, is odds. Odds can be expressed as the odds in favour of an event occurring or the odds against an event occurring. In sports it is actually more common to give the odds against something happening.

odds in favour
- ratio of the probability that an event will happen to the probability that it will not

odds against
- ratio of the probability that an event will not happen to the probability that it will

Odds

The odds in favour of $A = P(A) : P(A')$

The odds against $A = P(A') : P(A)$

Sports analysts often make predictions about a player's or team's chances for winning a tournament or championship. Often these predictions involve subjective probability based on the analyst's understanding of the player's or team's relative skill level.

Example 3

Odds of an Event

a) A hockey analyst gives the Canadian women's hockey team a 75% probability of winning the gold medal in the next Winter Olympics. Based on this prediction, what are the odds in favour of Canada winning Olympic gold?

b) A local sports journalist estimates that the high school boys' soccer team has a 40% probability of going to the OFSAA championship tournament. What are the odds against the boys making OFSAA?

Literacy Link

OFSAA, which stands for the Ontario Federation of School Athletic Associations, is a provincial association of student-athletes, teacher-coaches, and administrators.

Solution

a) The subjective probability of Canada winning the gold medal, $P(A)$, is given as 75%, or 0.75. The probability that Canada does not win is

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - 0.75 \\ &= 0.25 \end{aligned}$$

Use the definition of odds to calculate the odds of Canada winning gold.

$$\begin{aligned} \frac{P(A)}{P(A')} &= \frac{0.75}{0.25} \\ &= \frac{3}{1} \\ &= 3:1 \end{aligned}$$

How can you express as a ratio in lowest terms?

The odds in favour of the Canadian women's hockey team winning the gold medal at the next Winter Olympics are 3 : 1, based on the analyst's estimate.

b) Let A represent the event that the boys' soccer team makes it OFSAA. The probability that the boys' team goes to OFSAA, $P(A)$, is 40%, or 0.4. The probability that the team does not make OFSAA is

$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - 0.4 \\ &= 0.6 \end{aligned}$$

Calculate the odds against the boys' team making the OFSAA tournament using the definition

$$\begin{aligned} P(A'):P(A) &= 0.6:0.4 \\ &= 6:4 \\ &= 3:2 \end{aligned}$$

Reduce to lowest terms.

The odds against the high school boys' soccer team going to OFSAA are 3 : 2.

Your Turn

a) A sports commentator claims that the Toronto Raptors have a 60% probability of making the playoffs. Based on this estimate, what are the odds in favour of the Raptors making the playoffs?

b) It is estimated that a golfer has a 20% chance of winning a tournament. What are the odds against this golfer winning the tournament?

Consolidate and Debrief

Key Concepts

- The theoretical probability of an event occurring is a measure of its likelihood based on analysis of all possible outcomes.
- The theoretical probability of an event is calculated by dividing the total number of favourable outcomes by the total number of outcomes in the sample space.
- The probability of the complement of an event is the probability that the event will not occur.
- The odds in favour of an event is the ratio of the probability that the event will happen to the probability that it will not happen.
- The odds against an event occurring is the ratio of the probability that the event will not occur to the probability that it will.

Reflect

R1. a) Describe how the terms outcome, event, and sample space are related in terms of theoretical probability. Use a diagram, mind map, or other visual organizer to support your explanation.

b) Create an example that illustrates your answer to a).

R2. a) Explain how an event is related to its complement.

b) Create an example of an event and its complement, and determine their theoretical probabilities.

R3. a) What does odds in favour of an event mean?

b) What does odds against an event mean?

c) How are these concepts similar? How are they different?

Practise

Choose the best answer for #1 to #3.

1. Yuri is playing a card game. He will lose if he draws a face card (J, Q, or K) from a full deck of standard playing cards. What is the theoretical probability that Yuri will win his first draw?

A 6% **B** 9%

C 23% **D** 77%

2. A weather forecast predicts a 33% chance of rain tomorrow. What are the odds in favour of it raining tomorrow?

A 1:2 **B** 1:3

C 2:1 **D** 3:1

3. Susie is drawing toothpicks with four co-workers to see who will go on a snack run. What are the odds against Susie having to go get the snacks?

A 1:4 **B** 1:5

C 4:1 **D** 5:1

Apply

4. Two standard dice are thrown. Determine the theoretical probability that the sum is

a) 4 **b)** 7

c) an even number **d)** not a 6

e) not a perfect square

5. A drawer contains 3 black socks, 1 white sock, and 2 grey socks, all of the same style. Two socks are chosen from the drawer at random.

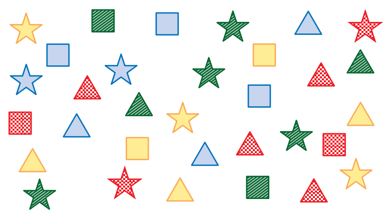
a) Describe the sample space using set notation.

b) Use set notation to show the different ways of choosing two socks that are the same colour.

c) What is the theoretical probability that two randomly chosen socks will be the same colour?

d) What are the odds in favour of two randomly chosen socks being the same colour?

6. In a lab study of learned behaviours, monkeys are taught to reach into a box and randomly choose a shape from the ones shown. If the monkey chooses any shape that is not red, he gets a reward. If he chooses a red shape, he gets nothing. By monitoring the monkey's behaviour as several trials are carried out, scientists can see whether there is any evidence that the monkey is able to recognize red.



a) Determine the theoretical probability that the monkey will be rewarded on any given trial, assuming that he randomly chooses a shape.

b) Determine the theoretical probability that the monkey in this example does not randomly choose a star.

7. **Communication** Refer to #6. Suppose that after several trials, the experiment is modified to allow the monkey to look into the box while choosing a shape. The table shows the results.

Colour Chosen	Count
Red	2
Not red	58

a) Determine the experimental probability that the monkey will choose a red shape.

b) Determine the experimental probability of the complementary event.

c) What might this suggest to the science researchers? Explain your reasoning.

8. Emily estimates that the odds against Paulo asking her to the prom are 4:1.

a) What type of probability is Emily applying?

b) What is the probability that Paulo will ask Emily to the prom, based on her estimate?

9. Chelsea is trying out for her school play. Using subjective probability, she estimates that she has an 80% chance of getting a part and a 25% probability of landing a lead role.

   a) What are the odds in favour of Chelsea getting a part in the play?

   b) What are the odds against her landing a lead role?

✓ Achievement Check

10. Kwon is answering four true or false questions on a quiz. Assume that he randomly guesses each answer.

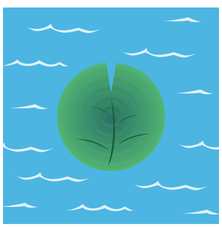
    a) Draw a tree diagram to illustrate all possible outcomes.

    b) What is the theoretical probability that he gets

    - all four correct?
    - exactly three correct?
    - fewer than two correct?
    - not all incorrect?

11. **Communication** Use a form of electronic (e.g., Internet) or print (e.g., newspaper) media to find an example in which the term “odds” is used. Describe the context and interpret the meaning of the information in terms of probability.

12. **Thinking** Puddles the frog is jumping around in Halls Lake. Sometimes she lands on a lily pad, and sometimes she falls into the water. Assume that on her next jump Puddles is equally likely to land anywhere in the square region shown below, including on top of the lily pad.



    a) Estimate the theoretical probability that Puddles will land on the lily pad.

    b) How many more times is she likely to fall into the water instead?

13. **Thinking** Suppose there are k possible outcomes to a certain probability experiment, all equally likely. Use algebraic reasoning to prove that the sum of the theoretical probabilities for all possible outcomes for this experiment must equal 1.

14. **Communication** A panel of hockey analysts gives 8 to 1 odds against the Montréal Canadiens winning the Stanley Cup and 17 to 2 odds against the Vancouver Canucks winning the Cup. Based on this information, which team is more likely to win the Stanley Cup? Explain your reasoning.

15. **Communication** A TV reporter states that “The chances of the Ottawa Senators winning against the Vancouver Canucks are 3:1 because they have won only one of their three meetings so far this year.”

    a) Describe the mathematical errors made by this reporter.

    b) Reword the statement so it is mathematically correct.

Extend

16. **Application**

    a) A fair coin is tossed n times. Determine an algebraic formula that will give the theoretical probability that all tosses will result in heads.

    b) Use the formula to determine the theoretical probability of tossing 10 heads in a row.

17. **Thinking**

    a) Explain how the odds in favour of an event occurring is related to the odds against the event occurring.

    b) Provide an example to illustrate your answer.

    c) Use algebraic reasoning to prove your answer to a).

1.3

Compare Experimental and Theoretical Probabilities

Learning Goal

I am learning to

- recognize the difference between experimental probability and theoretical probability

Minds On...

Have you ever wondered what it would be like to fly in space? Not many people get the opportunity to do that, but technology can be used to simulate the real experience. In fact, astronauts spend far more time preparing in a flight simulator than they actually do in space. What other types of simulators have you heard about?

Literacy Link

A *simulator* is a tool or machine that can be used to provide a sensation that is close to a real experience.

Action!

Technology can also be used to simulate a probability experiment, such as flipping a coin or rolling a pair of dice. This is particularly useful when carrying out a large number of trials. Flipping a coin 1000 times seems rather tedious when a graphing calculator or a computer program can simulate the task in a matter of seconds.

Investigate 1 Three-Coin-Flip Simulation

Materials

- coins
- graphing calculator with Probability Simulation application or
- computer with spreadsheet software

In this activity you will compare the experimental probability to the theoretical probability when three fair coins are flipped.

1. Draw a tree diagram to show all possible outcomes. Examine the sample space. How many possible outcomes are there?

2. What is the theoretical probability that there will be:

a) exactly three heads

b) exactly two heads

c) exactly one head

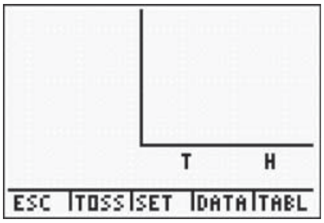
d) no heads

3. Choose one of the following three methods to conduct this simulation for a large number of trials.

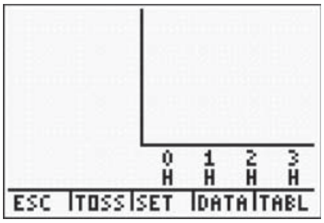
Method 1: Use a Graphing Calculator

a) • Press **APPS** and choose **Prob Sim**.
• From the main menu choose **1. Toss Coins**.

Note that you will be using the graphing calculator keys along the top row for a number of the menu choices.



b) • Choose **SET** by pressing the **ZOOM** key and change the number of coins from 1 to 3.
• Choose **OK**.



• Choose **TOSS** by pressing the **WINDOW** key to simulate one trial.

c) Repeat for a few trials and note how the results are tallied in the bar graph. Based on the theoretical probability values calculated earlier, create a sketch of what this bar graph should look like after a large number of trials are carried out.

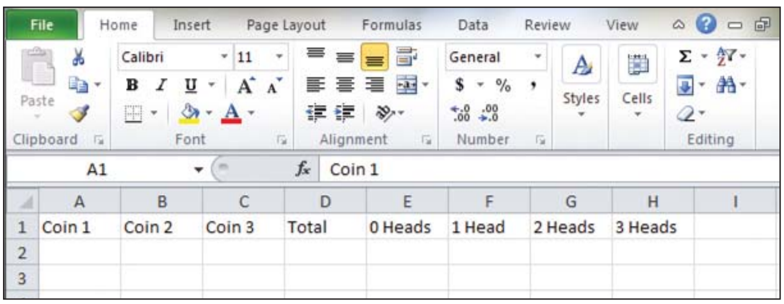
d) Carry out 100 trials. Here are some helpful tips:
• Use **+10** or **+50** to conduct 10 or 50 trials at a time.
• Use the left ⟵ and right ⟶ arrows to inspect the frequencies of each outcome.
• Choose **ESC** and then **TBL** to check the total number of trials carried out.
• Choose **GRPH** to return and conduct more trials.

How does the graph compare to the theoretical prediction?

e) Carry out 1000 trials. Describe how the bar graph changes.

Method 2: Use a Spreadsheet

a) Open a spreadsheet and label eight columns as shown.



Processes

Selecting Tools and Computational Strategies and Representing
In this simulation, the spreadsheet randomly generates an integer between 0 and 1.

A 1 represents heads, and a 0 represents tails.

What does the sum of any row of columns A to C represent?

b) Program the spreadsheet to simulate three coin tosses:
- Type the following in cell A2: =RANDBETWEEN(0,1)
- Copy the contents of cell A2 into cells B2 and C2.

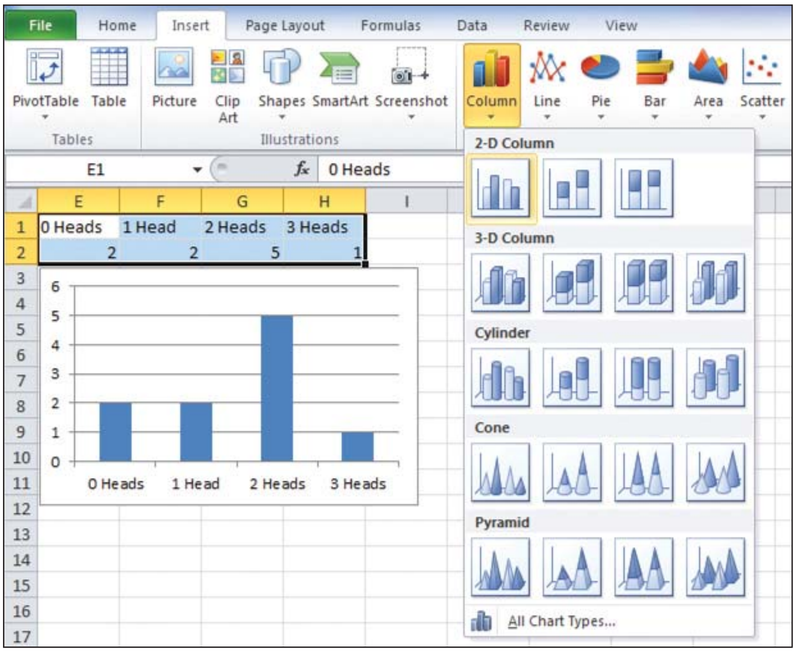
c) Program the spreadsheet to calculate the total number of heads for a given trial:
- Type the following in cell D2: =SUM(A2:C2)

d) Carry out 10 trials:
- Highlight cells A2 to D2.
- Drag the small square in the bottom right corner of D2 down to cell D11.

e) Program the spreadsheet to count the number of heads in each trial. For example, type "=COUNTIF(D:D,0)" into E2 to count how many trials included 0 heads. Use this command for counting 0, 1, 2, or 3 heads.
- Type the following in cell E2: =COUNTIF(D:D,0)
- Type the following in cell F2: =COUNTIF(D:D,1)
- Type the following in cell G2: =COUNTIF(D:D,2)
- Type the following in cell H2: =COUNTIF(D:D,3)

What do these commands do?

f) Create a bar graph to represent the frequency of each outcome:
- Highlight cells E1 to H2.
- Choose the **Insert** tab.
- From the **Column** menu, choose **Clustered Column**.



g) Carry out 100 trials:
- Highlight cells A11 to D11.
- Click and drag the bottom corner of cell D11 to cell D101.

How does the graph compare to the theoretical prediction?

h) Carry out 1000 trials. Describe how the bar graph changes.

Processes

Reflecting
Why is it necessary to go to row 101? Why not row 100?

Method 3: Combine Real Trials

a) Flip three coins 10 times. Determine the experimental probability of each outcome based on these 10 trials. Compare these values with the theoretical probabilities calculated in #2 above. How close are they?

b) Combine trials with your classmates until you have 100 trials in total. Sketch a bar graph of the results. How does the graph compare to the theoretical prediction?

c) Combine with your classmates' data from part b) to carry out 1000 trials and sketch a new graph. Describe how the bar graph changes as the number of trials increases.

4. **Reflect** Explain what happens to the statistical probabilities of this experiment as the total number of trials increases.

5. **Extend Your Understanding** Suppose this experiment were modified to include five coins, instead of three.

a) Do you think it would take more, fewer, or about the same number of trials for the statistical probabilities of the outcomes to match the theoretical probabilities? Explain your reasoning.

b) Design and carry out this experiment and write a brief report of your findings.

Investigate 2 Dice Simulation

Materials

- 2 dice
- graphing calculator with Probability Simulation application or
- computer with Fathom™

The table below illustrates all possible outcomes for the sum of the dice when two standard dice are thrown.

		Die 1					
	Outcome	⚀	⚁	⚂	⚃	⚄	⚅
Die 2	⚀	2	3	4	5	6	7
	⚁	3	4	5	6	7	8
	⚂	4	5	6	7	8	9
	⚃	5	6	7	8	9	10
	⚄	6	7	8	9	10	11
	⚅	7	8	9	10	11	12

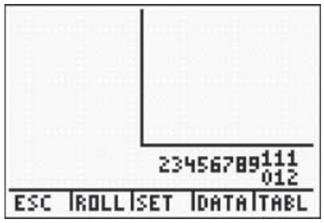
1. What is the theoretical probability of rolling each sum?

2. Sketch a bar graph showing the theoretical probability of rolling each sum.

3. Choose one of the following three methods to conduct this experiment for a large number of trials.

Method 1: Use a Graphing Calculator

a)
- Press APPS and choose **Prob Sim**.
- From the main menu choose **2. Roll Dice**.



b)
- Choose **SET** by pressing the ZOOM key and change the number of dice from 1 to 2.
- Choose **OK**.

c) Carry out more trials, 50 at a time, and describe how the graph is changing. Keep adding more trials until the statistical probabilities look very similar to the theoretical probabilities.
- Choose **ROLL** to carry out one trial.
- Use **+10** and **+50** to carry out several trials at a time.

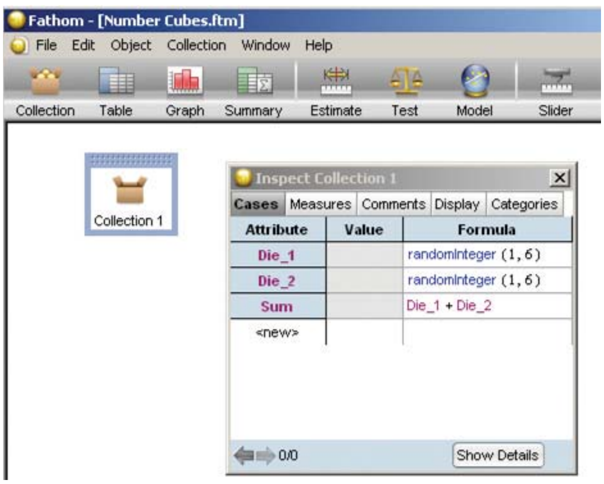
Method 2: Use Fathom™

a) Open Fathom™ and create a new collection by dragging the **Collection Box** into the workspace.

b) Create the dice simulation:
- Double click on **Collection 1** to open the **Inspector**.
- In the Attribute column, enter "Die_1", "Die_2", and "Sum" as shown.
- Double click under **Formula** and type the following for each die and then choose **OK**:
  randomInteger(1,6)

What do these commands do?

- Enter the formula for **SUM** as follows:
  - Click on **Attributes**.
  - Double click on **Die_1**.
  - Click on the + key.
  - Double click on **Die_2** and choose **OK**.



c) Simulate 10 trials:
- Right click on **Collection 1**.
- From the Collection menu, choose **New Cases**.
- Type 10 and choose **OK**.

Use the left and right arrows in the **Inspector** to view the outcomes.

d) Construct a bar graph to illustrate the outcomes:
- Click and drag a **Graph** into the workspace.
- Click on the **Sum** attribute in the **Inspector** and drag it onto the horizontal axis of the graph.
- Click on **Dot Plot** and change the graph to **Histogram**.

e) Conduct several trials. From the **Collection** menu, choose **New Cases**. Enter 90 and click **OK** to generate a total of 100 cases. Describe how the shape of the graph has changed.

f) Add new cases, 100 at time, and describe how the graph changes Keep adding cases until the statistical probabilities look very similar to the theoretical probabilities. How many cases did you have to use?

Literacy Link

A *histogram* shows the distribution of outcomes in an experiment. Although a histogram looks similar to a bar graph, it represents continuous data, so the bars are always placed side by side. You will learn more about histograms and their uses later in this course.

Method 3: Combine Real Trials

a) Roll two dice 10 times. Determine the experimental probability for each event. Are the statistical probabilities for these trials very useful? Explain why or why not.

b) Combine trials with your classmates until you have 100 trials in total. Sketch a bar graph of the results. How does the graph compare to the theoretical prediction?

c) Combine with your classmates' data from part b) to carry out 1000 trials and sketch a new graph. Describe how the bar graph is changing as the number of trials increases.

4. Reflect Explain what happens to the statistical probabilities of this experiment as the total number of trials increases.

5. Reflect Approximately how many trials did it take before the statistical probabilities closely agreed with the theoretical probabilities? Why do you think this is so?

6. Extend Your Understanding How would the outcomes of this experiment change if you used three dice instead of two? Design and carry out an experiment to find out. Describe your findings.

Project Prep

If your project involves probability, what elements of statistical, subjective, or theoretical probability could be relevant? Which tools and strategies could be useful for comparing experimental probability and theoretical probability?

Consolidate and Debrief

Key Concepts

- Probability experiments can be carried out using physical materials or technology-based simulations. Technology-based simulations are useful for carrying out very large numbers of trials.
- Experimental probability approaches theoretical probability as a very large number of trials are conducted.

Reflect

R1. Why is theoretical probability not a perfect predictor of the outcome of a probability experiment? Why is experimental probability not a perfect predictor of the outcome of a probability experiment?

R2. Why is it usually necessary to conduct a very large number of trials in order for experimental probability to reasonably agree with theoretical probability?

Practise

Choose the best answer for #1 and #2.

1. A die is rolled once and turns up a 4. Which statement is true about rolling a 3?

A The experimental probability is 0 and the theoretical probability is $\frac{1}{6}$.

B The experimental probability is 1 and the theoretical probability is $\frac{1}{6}$.

C The experimental probability is $\frac{1}{6}$ and the theoretical probability is $\frac{1}{6}$.

D The experimental probability is 1 and the theoretical probability is 1.

2. Which of the following statements is true?

A Experimental probability is always equal to theoretical probability.

B Experimental probability approaches theoretical probability when a very large number of trials are carried out.

C Experimental probability is always a more reliable predictor than theoretical.

D Theoretical probability is always a more reliable predictor than experimental.

Apply

3. Application

a) Use the *Spin Spinner* simulation in the graphing calculator Probability Simulator to construct a mystery spinner:
- From the main menu, choose **4. Spin Spinner**.
- Choose **SET**. Design your spinner
- Choose **ADV**. Change the probability values. Note that the sum of these values must equal 1.
- Choose **OK** twice.

b) Trade with a classmate. Work with your partner's mystery spinner. Look at the spinner. Estimate the theoretical probability of each outcome.

c) Carry out several trials to determine the experimental probability of each outcome. When you are confident that you have solved the mystery spinner, check the spinner's design by choosing **SET** and then **ADV**.

4. **Application**
   a) Use the *Pick Marbles* simulation in the graphing calculator Probability Simulator to construct a mystery bag of marbles.
      - From the main menu, choose **3. Pick Marbles**.
      - Choose **SET**. Design your mystery bag.
   b) Trade with a classmate. Try to guess each other's mystery bag using experimental probability.

5. Suppose you rolled two 8-sided dice, each having values from 1 to 8 on their faces.
   a) What sums are possible? What is the theoretical probability of rolling each sum?
   b) Sketch a bar graph showing the theoretical probability of rolling each sum.
   c) Conduct a large number of trials of this probability experiment using a simulation tool or strategy of your choice. Discuss how the statistical and theoretical probabilities compare over:
      - a few trials
      - a very large number of trials
   d) About how many trials did it take for the statistical and theoretical probability values to agree closely?

✓ Achievement Check

6. Two fair coins are tossed at the same time.
   a) Identify the sample space that represents all possible outcomes.
   b) Determine the theoretical probability of
      - no heads
      - exactly one head
      - two heads
   c) Draw a bar graph that predicts how the statistical probabilities for a large number of trials will look.
   d) Use a technology-based simulation tool to carry out a very large number of trials.
   e) Compare the resulting graph to the one you drew in c). How close are they? How many trials did you use to get them to look very similar?

7. **Open Question**
   a) Design a probability experiment involving coins and/or dice. Determine the theoretical probability of each event.
   b) Use a simulation tool or other strategy to carry out a very large number of trials.
   c) Describe the trend behaviour of the statistical probabilities as the number of trials increases.

8. **Thinking** Two dice are thrown and the sum is recorded.
   a) If 20 trials are carried out, is it possible for the statistical probabilities of each value to be equal to the corresponding theoretical probabilities? Explain why or why not.
   b) What is the minimum number of trials necessary for statistical and theoretical probability values to perfectly agree in this case? Justify your answer.

Extend

9. **Thinking**
   a) Choose one of the graphing calculator probability simulations that was not explored in this section. Briefly describe what it simulates.
   b) Create and solve a probability problem that can be done with this simulation.
   c) Trade with a classmate and solve each other's problems.

10. **Open Question**
   a) Find an online probability simulator on the Internet. Briefly explain what it does.
   b) What are some things you like about this simulator?
   c) What are some things you wish it did better?

1.4

Mutually Exclusive and Non-Mutually Exclusive Events

Learning Goal

I am learning to

- describe how an event can represent a set of probability outcomes
- recognize how different events are related
- calculate the probability of an event occurring

Minds On...

Playing cards has been a popular source of entertainment for hundreds of years. Simple games such as Snap and Crazy Eights can be easily learned by young children. More complex strategy-based games such as bridge can be more challenging.

- What are some card games that you have heard of?
- What makes them interesting or fun to play?
- How are the games related to probability?

Literacy Link

Unless otherwise noted, we will use the term "probability" to refer to *theoretical* probability from this point onward.

Action!

Investigate Counting Cards

Materials

- standard deck of playing cards
- Venn diagram

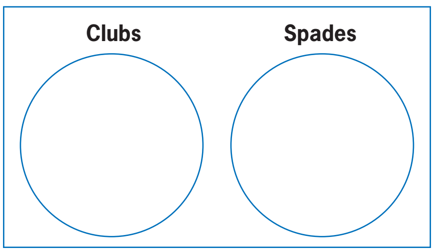
A standard deck of playing cards is represented below.

A♣ 2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣
A♦ 2♦ 3♦ 4♦ 5♦ 6♦ 7♦ 8♦ 9♦ 10♦ J♦ Q♦ K♦
A♠ 2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠
A♥ 2♥ 3♥ 4♥ 5♥ 6♥ 7♥ 8♥ 9♥ 10♥ J♥ Q♥ K♥

1. a) How many cards are clubs?
   b) How many cards are spades?

2. Add the answers to 1a) and 1b).

3. Create a Venn diagram by placing or listing all of the cards that are either a club or a spade.



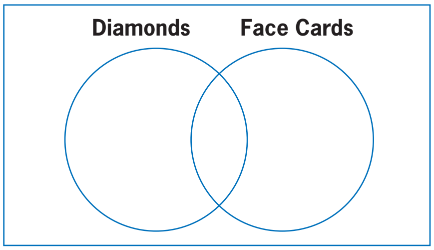
4. How many total cards are in the Venn diagram?

5. **Reflect** Are the answers to step 2 and step 4 the same? Explain why or why not.

6. a) How many cards are diamonds?
   b) How many cards are face cards?

7. Add the answers to 6a) and 6b).

8. Create a Venn diagram by placing or listing all of the cards that are a diamond, a face card, or both.



9. How many total cards are in the Venn diagram?

10. **Reflect** Are the answers to step 7 and step 9 the same? Explain why or why not.

11. **Extend Your Understanding**
    a) Determine the probability of randomly drawing either a club or a spade from a standard deck of cards.
    b) Determine the probability of randomly drawing either a diamond or a face card.
    c) Explain your methods.

Processes

Selecting Tools and Computational Strategies
How is a Venn diagram useful for comparing the attributes of cards?

Literacy Link

Unless otherwise noted, we will use the term "probability" to refer to *theoretical* probability from this point onward.

When calculating the probability of either one event happening or another happening, it is important to carefully count the outcomes. This is relatively easy to do when the events have completely different characteristics. Such events are said to be **mutually exclusive events**, because you can have either one event or the other, but not both.

When you flip a coin, it comes up either heads or tails. It cannot be both. Similarly, a newborn is either a boy or a girl. When a card is drawn from a standard deck of cards, it will be a club, a diamond, a heart, or a spade. All of these are examples of mutually exclusive events.

mutually exclusive events
- events that have different attributes
- cannot occur simultaneously

Example 1

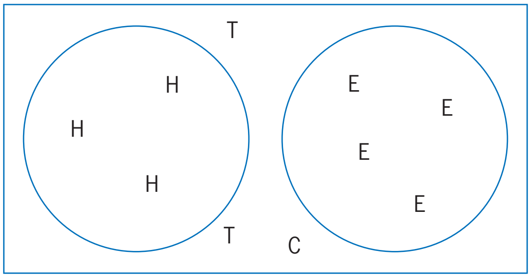
Probability of Mutually Exclusive Events

A picnic basket of sandwiches contains 3 ham, 2 turkey, 1 chicken, and 4 egg salad sandwiches. What is the probability of reaching in and randomly choosing either a ham or an egg salad sandwich?

Solution

Method 1: Examine the Sample Space

Use a Venn diagram to illustrate all possible outcomes. The favourable outcomes are H or E.



There are 7 out of 10 favourable outcomes. The probability of randomly choosing a ham or an egg salad sandwich is

$$\begin{aligned} P(H \text{ or } E) &= \frac{7}{10} \\ &= 70\% \end{aligned}$$

Method 2: Add the Probabilities

Determine the probability of each favourable event and add them together. There are 3 ham sandwiches and 4 egg salad sandwiches, out of a total of 10.

$P(H) = \frac{3}{10}$ and $P(E) = \frac{4}{10}$

Add the probabilities.

$$\begin{aligned} P(H) + P(E) &= \frac{3}{10} + \frac{4}{10} \\ &= \frac{7}{10} \\ &= 70\% \end{aligned}$$

Therefore, the probability of randomly drawing a ham or an egg salad sandwich is 70%.

Your Turn

A cooler contains the following juice bottles: 3 orange, 5 apple, 3 citrus blend, and 4 grape. What is the probability of reaching in and randomly choosing an apple or grape juice?

The second method used in Example 1 shows the additive principle for mutually exclusive events. You can use algebraic reasoning to show that this method is valid for any such situation. In Method 1, all favourable events were counted and divided by the total number of possible outcomes:

$$
\begin{aligned}
P(A \text{ or } B) &= \frac{n(A \text{ or } B)}{n(S)} \\
&= \frac{n(A) + n(B)}{n(S)} \\
&= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} \\
&= P(A) + P(B)
\end{aligned}
$$

In how many ways can A or B occur?

Write as separate fractions and apply the definition of probability.

This result represents the approach that was used in Method 2.

Additive Principle (Rule of Sum) for Mutually Exclusive Events

The probability of either of two mutually exclusive events, A or B, is:

$P(A \text{ or } B) = P(A) + P(B)$

Literacy Link

In this resource, we will call the additive principle the *rule of sum.*

Example 2

Apply the Rule of Sum for Mutually Exclusive Events

A number of actors have starred as James Bond over the years. The table summarizes Rolly's Bond movie collection, tallied by actor.

Actor	Number of Movies
Sean Connery	6
George Lazenby	1
Roger Moore	7
Timothy Dalton	2
Pierce Brosnan	4
Daniel Craig	2

Sometimes Rolly likes to pick a Bond movie to watch at random. What is the probability that Rolly will randomly pick either a Connery, C, or a Dalton, D, film from his shelf?

Solution

There are two favourable outcomes. Determine their probabilities and apply the rule of sum for mutually exclusive events. There are 22 movies in total. Therefore:

$$P(C) = \frac{n(C)}{n(S)} \qquad P(D) = \frac{n(D)}{n(S)}$$
$$= \frac{6}{22} \qquad\qquad = \frac{2}{22}$$

Why should these fractions be left unreduced?

Add the probabilities of the two favourable outcomes:

$$\begin{aligned} P(C \text{ or } D) &= P(C) + P(D) \\ &= \frac{6}{22} + \frac{2}{22} \\ &= \frac{8}{22} \\ &= 0.3636\ldots \end{aligned}$$

The probability of Rolly randomly picking a Connery or a Dalton movie is approximately 36%.

Your Turn

What is the probability that Rolly will randomly pick either a Brosnan or a Moore movie?

In some situations, it is possible for different events to occur simultaneously. For example, in a standard deck of cards, let $n(D)$ represent the number of diamonds and $n(F)$ represent the number of face cards.

A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥

From the diagram, $n(D) = 13$ and $n(F) = 12$. Adding these gives

$$\begin{aligned} n(D) + n(F) &= 13 + 12 \\ &= 25 \end{aligned}$$

This sum includes three cards that belong to both sets, J♦, Q♦, and K♦, which have been counted twice. This represents an example of **non-mutually exclusive events**.

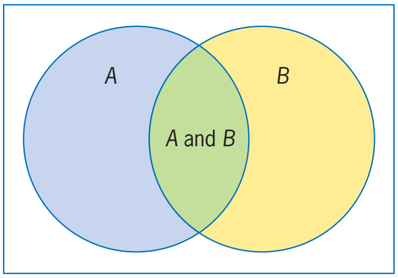
non-mutually exclusive events
- different events that can happen at the same time

A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♠	2♠	3♠	4♠	5♠	6♠	7♠	8♠	9♠	10♠	J♠	Q♠	K♠
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥

To obtain the correct number of diamonds and face cards, subtract the three cards that were counted twice from the sum found above:

$25 - 3 = 22$

This counting strategy is known as the principle of inclusion and exclusion.



Principle of Inclusion and Exclusion

If A and B are non-mutually exclusive events, then the total number of favourable outcomes is:

$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$

Literacy Link

When counting outcomes from different sets, the following notation is sometimes used:

The *union set*, $n(A \cup B)$, is the combined set of elements contained in either sets A or B.

The *intersection set*, $n(A \cap B)$, is the overlapping set of elements contained in both sets A and B.

Example 3

Principle of Inclusion and Exclusion

In a room with 30 students, 10 play basketball and 15 play volleyball. If 7 students play both sports, what is the probability that a student chosen at random plays basketball or volleyball?

Solution

Apply the principle of inclusion and exclusion to count the number of students who play basketball $n(B)$ or volleyball $n(V)$. Then divide by the total number of students, $n(S)$.

$$\begin{aligned} n(B \text{ or } V) &= n(B) + n(V) - n(B \text{ and } V) \\ &= 10 + 15 - 7 \\ &= 18 \end{aligned}$$

The probability of randomly choosing a basketball or volleyball player is

$$\begin{aligned} P(B \text{ or } V) &= \frac{n(B \text{ or } V)}{n(S)} \\ &= \frac{18}{30} \\ &= \frac{3}{5} \end{aligned}$$

Therefore, there is a $\frac{3}{5}$, or 60%, probability of choosing a basketball player or volleyball player.

Your Turn

Miranda is part of a gift exchange with 24 other family members and friends. Of the 24 family members, 10 like to ski, 12 like to cycle, and 6 like to ski and cycle. If Miranda randomly draws a name, what is the probability that she will pick someone who likes to ski or cycle?

To calculate the probability of two non-mutually exclusive events, A and B, add the probability of each event and subtract the probability of both events occurring simultaneously.

Probability of Non-Mutually Exclusive Events

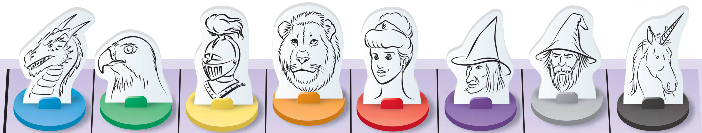
The probability of either of two non-mutually exclusive events, A or B, is:

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example 4

Probability of Non-Mutually Exclusive Events

The playing tokens and their characteristics for a role-playing game are shown below.



	Dragon	Hawk	Knight	Lion	Princess	Witch	Wizard	Unicorn
Human			✓		✓	✓	✓	
Animal	✓	✓		✓				✓
Supernatural	✓					✓	✓	✓
Can fly	✓	✓				✓		
Can cast spells						✓	✓	

If Jozo is randomly assigned a playing token, what is the probability that it will be either an animal or a supernatural creature?

Solution

Calculate the probability that Jozo randomly picks an animal or a supernatural creature by applying either the probability of non-mutually exclusive events or the principle of inclusion and exclusion.

There are 4 animals: dragon, hawk, lion, and unicorn. There are 8 tokens in total.

$$P(A) = \frac{4}{8}$$
$$= \frac{1}{2}$$

There are 4 supernatural creatures: dragon, witch, wizard, and unicorn.

$$P(S) = \frac{4}{8}$$
$$= \frac{1}{2}$$

There are 2 supernatural animals: dragon and unicorn.

$$P(A \text{ and } S) = \frac{2}{8}$$
$$= \frac{1}{4}$$

The probability that Jozo will randomly choose an animal or a supernatural creature is

$$P(A \text{ or } S) = P(A) + P(S) - P(A \text{ and } S)$$
$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$

Therefore, Jozo has a $\frac{3}{4}$, or 75%, probability of randomly picking an animal or a supernatural creature.

Your Turn

What is the probability that Jozo will randomly pick a flying creature or one that can cast spells?

Consolidate and Debrief

Key Concepts

- Mutually exclusive events cannot occur at the same time.
- To calculate the probability of either mutually exclusive events A or B occurring, use the additive principle (rule of sum) for mutually exclusive events:

  $$P(A \text{ or } B) = P(A) + P(B)$$

- Non-mutually exclusive events can occur at the same time.
- To calculate the number of outcomes included in non-mutually exclusive events A and B, use the principle of inclusion and exclusion:

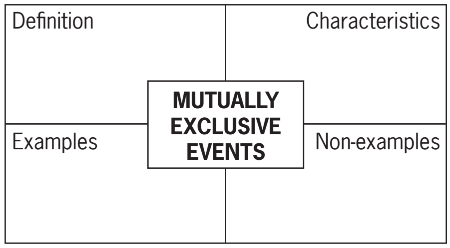
  $$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$$

- To calculate the probability of either non-mutually exclusive events A or B occurring, use the additive principle for non-mutually exclusive events:

  $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Reflect

R1. Copy and complete the Frayer model shown for the term "mutually exclusive events."



R2. **a)** What is the principle of inclusion and exclusion?

b) When and why is it important to use it?

R3. Provide an example of non-mutually exclusive events that are different from those already shown.

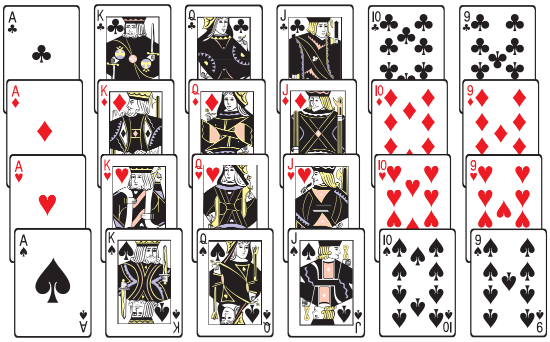
Practise

Choose the best answer for #1 and #2.

1. What is the probability of rolling a 3 or 4 using a standard die?

A $\frac{1}{6}$ **B** $\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$

2. The card game euchre uses only the cards shown from a standard deck of playing cards.



What is the probability of randomly drawing an ace or a king from a euchre deck of cards?

A $\frac{5}{12}$ **B** $\frac{1}{2}$ **C** $\frac{7}{12}$ **D** $\frac{1}{3}$

Apply

3. **Communication** Kara's shirt collection is shown below.

Her shirts are jumbled in a drawer.

a) Determine the probability that Kara randomly draws each of the following:
- a pink shirt or a purple shirt
- a pink shirt or a short-sleeved shirt

b) Which of the scenarios in a) represent:
- a mutually exclusive event?
- a non-mutually exclusive event?

Explain your answers.

4. **Application** Every Friday night, Rutger's family orders take-out. The table shows their ordering habits for the past several weeks.

Type of Food	Tally
Pizza	𝍸
Mexican	𝍷𝍷
Burgers	𝍷𝍷𝍷𝍷
Chicken	𝍷𝍷𝍷

Rutger's favourites are Mexican and chicken. What is the experimental probability that Rutger will get one of his favourites next Friday?

5. What is the probability of rolling a sum that is not a 7 or an 11 with a pair of dice?

6. Refer to the euchre deck of cards in #2.

a) Determine the probability of randomly drawing either an ace or a spade from the deck.

b) What is the probability of randomly drawing a red card or a diamond from the deck?

c) What is the probability of not drawing a face card or a club?

7. Refer to the euchre deck of cards in #2.

a) What is the probability of randomly drawing a 9 or a 10 or a diamond from the deck?

b) Explain how you solved this problem.

8. **Thinking** Deer Button is a game played by people of the Woodland Nations. Players use eight two-colour counters made from deer's antlers, like the ones shown below.

Players take turns throwing all eight deer buttons at the same time. They win beans according to this scoring table:

Number of Buttons of the Same Colour	Beans Awarded
8	10
7	4
6	2
other	0

a) Determine the probability that a player will score 10 points on a given throw.

b) What is the probability of scoring at least 4 points on a throw?

c) Explain how you solved this problem.

✓ Achievement Check

9. Juliette puts these letter tiles into her handbag.

a) If Juliette then reaches into the handbag and randomly takes out one tile, determine the probability of each of the following occurring:

- She chooses an "e" or a "t."
- She chooses a red letter or an "e."
- She chooses a capital letter or a vowel.
- She does not choose a yellow letter or a "t."

b) Draw a Venn diagram to represent each scenario in part a).

c) **Open Question** Create a probability question using these tiles for which the answer is between 25% and 40%.

10. **Open Question** A bag contains three blue marbles and some other marbles. There is a 50% probability that a randomly chosen marble is either green or yellow.

a) What could the contents of the bag be?

b) Provide a different answer that is also correct.

11. **Thinking** Marie is playing a board game and can win if she rolls a sum of either 6 or 8 or doubles with a standard pair of dice. What are the odds against Marie winning on a given throw?

12. **Open Question** Create and solve a probability problem involving mutually exclusive events.

Extend

13. Use algebraic reasoning to prove that the probability of two non-mutually exclusive events, A and B, can be calculated using $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

14. **Application** Renzo knows that his first-semester timetable will include biology, chemistry, English, and a study period, but he does not know when each will occur during the day. Two periods run in the morning and two periods run in the afternoon. The time of day for each course does not change.

a) What is the probability that Renzo will have both science classes in the morning or both in the afternoon?

b) Explain how you solved this problem.

c) Discuss any assumptions you made in your solution.

15. **Thinking**

a) Use algebraic reasoning to develop the additive principle for three non-mutually exclusive events.

b) **Open Question** Design a question that can be solved using the result from part a). Then solve the problem.

1.5

Independent and Dependent Events

Learning Goal

I am learning to

- describe and determine how the probability of one event occurring can affect the probability of another event occurring
- solve probability problems involving multiple events

Minds On...

Modern technology allows doctors to predict the gender of a baby long before it is born. Some parents like to know this information as soon as possible, while others prefer to keep it a surprise.

Look at the family pictured.

- Do you think having three boys and three girls is very likely?
- Do you think the gender of one birth will have an effect on the gender of a following birth? Why or why not?

When multiple events occur in a probability experiment, these events are called compound events.

compound events

- multiple events in a probability experiment
- may or may not affect each other

Action!

Example 1

Simple Compound Events

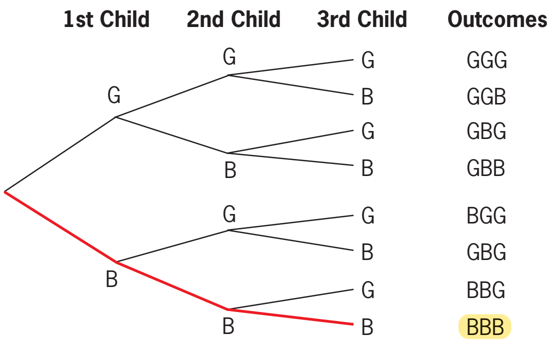
a) The Archers currently have no children, but would like to have three children. Assuming that they do have three children and there is an equal probability of any birth resulting in a boy or a girl, what is the probability that they will all be boys?

b) The Burnell family has two sons. What is the probability that their third child will be a boy?

c) Why are the answers to a) and b) different?

Solution

a) The sample space for the Archers is shown.

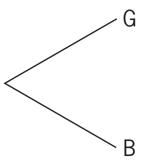


The tree diagram shows that there are eight possible outcomes, only one of which results in three boys. The probability of the Archers having three boys is

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{1}{8}$$

The probability that the Archers will have three boys is $\frac{1}{8}$, or 12.5%, assuming that they have three children.

b) Assume that there is an equal probability of any birth resulting in a boy or a girl. The fact that the Burnells already have two sons has no impact on the outcome of their third birth. The sample space for the gender of the third child is shown.



Therefore, the probability that the third Burnell child will be a boy is 50%.

c) The answers to a) and b) are different because they represent different situations. In the case of the Archers, the probability of multiple events is being considered. The Burnell case involves just a single event.

Your Turn

The Singh family currently has no children but hopes to have four children. Assuming they do have four children, what is the probability that they will all be girls?

Processes

Selecting Tools and Computational Strategies
A tree diagram is a useful way to clearly identify and organize all possible outcomes.

Sometimes the probability of one event depends on whether another event occurs, and sometimes it does not. When one event has no effect on the probability of another, the events are **independent**.

independent events
- situations in which the occurrence or non-occurrence of one event has no influence on the probability of the other event occurring

Example 2

Probability of Independent Events

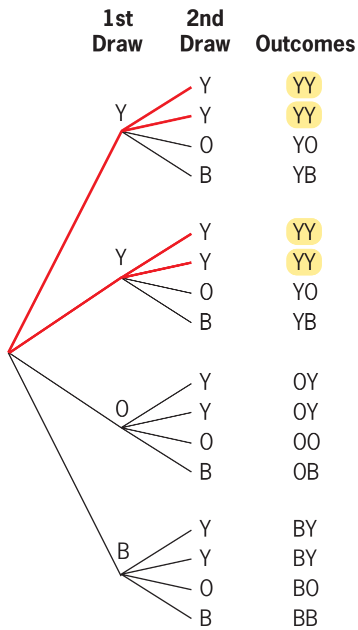
Olivia has four highlighting pens in her pencil case: two yellow, one orange, and one blue. She reaches into her pencil case and randomly chooses a highlighter. After she uses it, she immediately replaces it in the case so it can be used again. What is the probability that she will choose

a) two yellow highlighters in a row?

b) a yellow highlighter followed by a blue highlighter?

Solution

a) Use a tree diagram to illustrate all possible outcomes. Since the highlighters are replaced into the case immediately after being used, these are independent events.

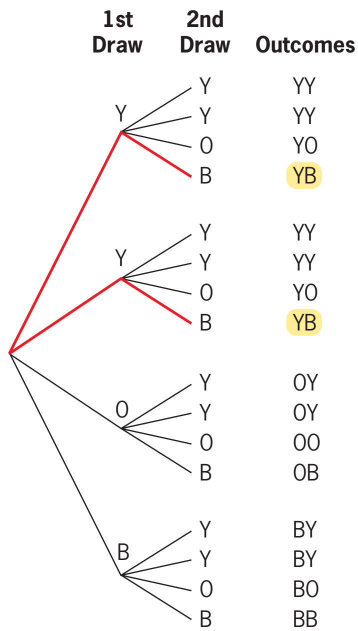


Olivia can choose two yellow highlighters in a row 4 out of 16 ways. Therefore,

$$P(YY) = \frac{4}{16}$$
$$= \frac{1}{4}$$

The probability of Olivia choosing two yellow highlighters in a row is $\frac{1}{4}$.

b) Inspect the tree diagram and identify the ways in which a yellow highlighter can be followed by a blue highlighter.



Olivia can choose a yellow highlighter followed by a blue highlighter in a row 2 out of 16 ways. Therefore,

$$P(YB) = \frac{2}{16}$$
$$= \frac{1}{8}$$

The probability of Olivia choosing a yellow highlighter followed by a blue highlighter is $\frac{1}{8}$.

Your Turn

Three green marbles and two yellow marbles are placed into a bag. What is the probability of randomly drawing a green marble followed by a yellow marble, assuming that the first marble is replaced before the second marble is drawn?

Another way to determine the probability of compound independent events is to multiply the probability of the first event by the probability of the second event. In Example 2, the probability of Olivia choosing a yellow highlighter is 2 out of 4 or $\frac{1}{2}$, and the probability of choosing a blue highlighter is 1 out of 4 or $\frac{1}{4}$.

Using this method, the probability of choosing two yellow highlighters in a row is

$$\begin{aligned} P(YY) &= \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Similarly, calculate the probability of choosing a yellow highlighter followed by a blue highlighter.

$$\begin{aligned} P(YB) &= \frac{1}{2} \times \frac{1}{4} \\ &= \frac{1}{8} \end{aligned}$$

Note that these values agree with the ones obtained from analysing the tree diagram.

This result is an illustration of the multiplicative principle for independent events.

Multiplicative Principle (Fundamental Counting Principle) for Independent Events

The probability of two independent events, A and B, occurring is

$P(A \text{ and } B) = P(A) \times P(B)$

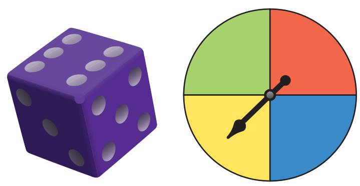
Literacy Link

In this resource, we will refer to the multiplicative principle as the *fundamental counting principle*. You will learn more about this principle in a later chapter.

Example 3

Different Compound Events

A game is played in which a standard die is rolled and a spinner is spun.



Player A wins a point if the spinner lands on red and an even number is rolled. Player B wins a point if the spinner lands on yellow or green and a composite number is rolled. Is this a fair game? Explain.

Solution

Assume that the spinner and the die are fair. Determine the probability of each player's winning compound event.

Player A wins a point if the spinner lands on red, R, and the die turns up even, E. Since one-quarter of the spinner area is red,

$$P(R) = \frac{1}{4}$$

Since three of six possible outcomes for a standard die are even,

$$\begin{aligned} P(E) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

The spinner and die results are independent events.

The compound probability of Player A winning a point, $P(A)$, is

$$\begin{aligned} P(A) &= P(R \text{ and } E) \\ &= P(R) \times P(E) \\ &= \frac{1}{4} \times \frac{1}{2} \\ &= \frac{1}{8} \end{aligned}$$

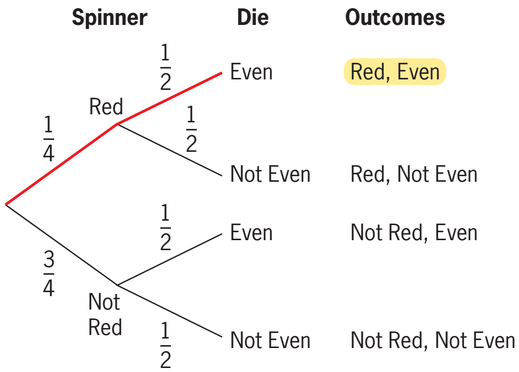
Player A has a $\frac{1}{8}$, or 12.5%, probability of winning a point on a given trial.

Processes

Selecting Tools and Computational Strategies

How can you use a probability tree diagram to efficiently calculate the probability of compound events?

You can use a probability tree diagram to illustrate this solution.

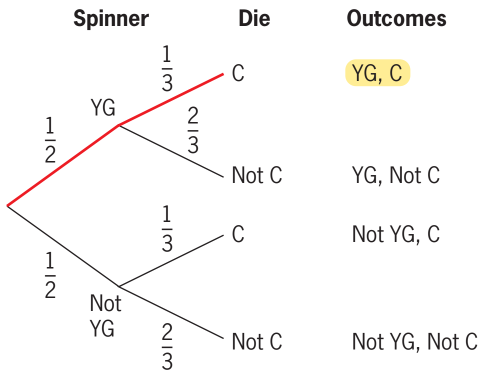


Player B wins a point if the spinner lands on yellow or green, YG, and the die turns up a composite number, C. Since half of the spinner area is yellow or green,

$$P(YG) = \frac{1}{2}$$

There are two composite numbers on a die: 4 and 6. The probability of rolling a composite number is

$$\begin{aligned} P(C) &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$



Since the spinner and die results are independent events, the compound probability of Player B winning a point, $P(B)$, is

$$\begin{aligned} P(B) &= P(YG \text{ and } C) \\ &= P(YG) \times P(C) \\ &= \frac{1}{2} \times \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

Therefore, Player B has a $\frac{1}{6}$, or approximately 17%, probability of winning a point on a given trial. This game is not fair. Player B has an advantage over Player A, who has just a 12.5% probability of winning a point.

Your Turn

What is the probability of flipping heads with a fair coin and rolling a prime number with a fair die?

Sometimes the probability of one event occurring has an effect on another event occurring. When this happens, the events are said to be dependent.

dependent events

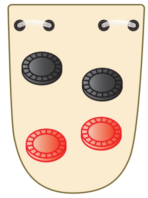
- the occurrence or non-occurrence of one event influences the probability of the other event occurring

Example 4

Probability of Dependent Events

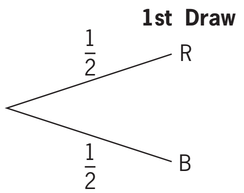
Two red checkers and two black checkers are placed into a bag. What is the probability that a red checker is randomly chosen, followed by a second red checker, assuming that the first checker drawn is not replaced?

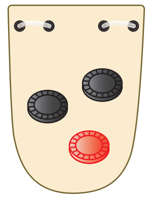
Solution



Let $P(R)$ represent the probability of choosing a red checker and $P(RR)$ represent the probability of choosing two consecutive red checkers. Two out of four checkers are red.

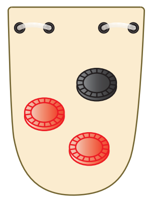
Therefore, $P(R) = \frac{1}{2}$.





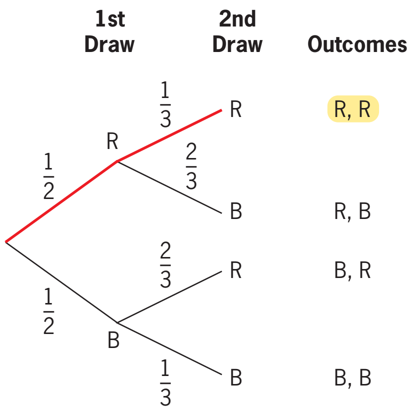
After the first checker is chosen, there will be three left.

In this case, the probability of choosing a second red checker, given that the first checker chosen was red, is $\frac{1}{3}$.



If the first checker chosen is black, then the bag will have three checkers.

In this case, given that the first checker chosen was black, the probability of drawing a red checker is $\frac{2}{3}$. Use these values to complete the probability tree diagram.



To determine the probability of randomly choosing two consecutive red checkers, multiply the probabilities along the RR branch.

$$P(RR) = \frac{1}{2} \times \frac{1}{3}$$
$$= \frac{1}{6}$$

Therefore, there is a $\frac{1}{6}$, or approximately 17%, probability of randomly choosing a red checker followed by another red checker.

Your Turn

A bag contains two apples, one orange, and two peaches. Suppose Jelena reaches in and chooses a piece of fruit at random, and then selects another piece of fruit without replacing the first one. What is the probability that she will choose two peaches?

The previous example illustrates how the outcome of one trial can affect the probable outcome of a subsequent trial. When the first checker was chosen and not replaced the sample space was reduced, which changed the probability values for the second trial.

When the outcome of one trial has been determined, then the **conditional probability** of a subsequent trial can be calculated based on the result of the first trial. If the first event is A and the second event is B, then $P(B|A)$ represents the conditional probability that B will occur, given that A has occurred. To calculate the probability of both dependent events occurring, apply the multiplicative principle for dependent events.

conditional probability

- probability of a second event occurring, given that a first event occurred
- the sample space for the second event is reduced from the first event

Multiplicative Principle for Dependent Events

To calculate the probability of two dependent events, A and B, occurring, multiply the probability that A occurs by the conditional probability that B occurs, given that A occurred.

$P(A \text{ and } B) = P(A) \times P(B|A)$

Example 5

Conditional Probability in Telemarketing

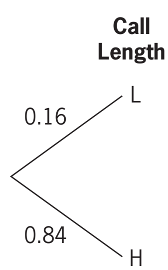
A study of a telemarketing company's data showed that of 1000 calls placed:

- The experimental probability of a call receiver staying on the line for at least a minute was 16%.
- The conditional probability of a call resulting in a sale, given that a receiver stayed on the line for at least a minute, was 10%.
- No sales were made if the receiver did not stay on the line for at least a minute.

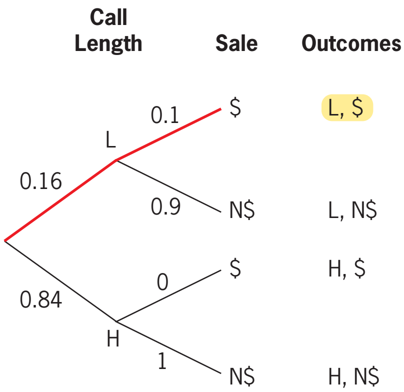
How many sales were made?

Solution

Construct a probability tree diagram to represent the data. There is a 16% experimental probability of having a long call, L, and therefore an 84% chance the caller will hang up, H.



If the caller hangs up early, the probability of a sale is 0. If the caller stays on the line, there is a 10% experimental probability of a sale. Add this information to the tree diagram.



Multiply the experimental probability that a customer stays on the line by the conditional probability that the call results in a sale, given that the customer stays on the line.

$$\begin{aligned} P(L \text{ and } \$) &= P(L) \times P(\$|L) \\ &= 0.16 \times 0.1 \\ &= 0.016 \end{aligned}$$

Therefore, there is a 0.016, or 1.6%, experimental probability that a random call will result in a sale. Use the definition of experimental probability to calculate the number of sales.

$$P(\$) = \frac{n(\$)}{n(S)}$$

where $n(\$)$ is the number of sales made and $n(S)$ is the total number of calls made. Solve for the unknown, $n(\$)$.

$$P(\$) \times n(S) = \frac{n(\$)}{n(S)} \times n(S)$$ Multiply both sides by $n(S)$.

$$n(\$) = P(\$) \times n(S)$$ Substitute and solve.

$$\begin{aligned} &= 0.016 \times 1000 \\ &= 16 \end{aligned}$$

Therefore, 16 sales were made from the 1000 calls that were studied.

Your Turn

Lars is offering juice samples at a shopping mall. The experimental probability of a randomly chosen shopper accepting a sample is 15%. The conditional probability of a customer purchasing some juice given that he or she tried a sample is 20%. No one purchases juice without trying a sample. If Lars offers 500 people juice samples, how many sales will he make?

Consolidate and Debrief

Key Concepts

- Compound events involve more than one event for a given trial of a probability experiment.
- Independent events have no influence on each other's probability of occurring.
- To calculate the probability of two independent events, A and B, both occurring, multiply the probability of each of them occurring:

  $P(A \text{ and } B) = P(A) \times P(B)$
- When the occurrence or non-occurrence of one event influences the probability of a second event occurring, the events are dependent.
- To calculate the probability of two dependent events, A and B, both occurring, multiply the probability of the first event occurring by the conditional probability of the second event occurring, given that the first event occurred:

  $P(A \text{ and } B) = P(A) \times P(B|A)$

Reflect

R1. **a)** What is the difference between independent events and dependent events?

b) Provide an example of each.

R2. Which of the following scenarios is most likely to occur, and why?
- a coin is flipped three times and comes up heads every time
- after coming up heads four times, a coin comes up heads on the fifth toss

R3. **a)** Explain what is meant by conditional probability.

b) Describe a situation in which conditional probability
- applies
- does not apply

R4. What are some advantages of using a probability tree diagram in solving problems involving dependent events?

Practise

Choose the best answer for #1 to #3.

1. A fair coin is flipped twice. What is the probability that it will come up heads followed by tails?

A 0 **B** $\frac{1}{8}$ **C** $\frac{1}{4}$ **D** $\frac{1}{2}$

2. Hanna forgot to study for her math quiz. In the multiple-choice section there are two questions, each with four answer choices. If Hanna randomly guesses the answer to both questions, what is the probability that she will get them both correct?

A 0% **B** 6.25% **C** 12.5% **D** 25%

3. A fair coin is flipped twice. What is the probability that it will come up once heads and once tails, in either order?

   A 0

   B $\frac{1}{8}$

   C $\frac{1}{4}$

   D $\frac{1}{2}$

Apply

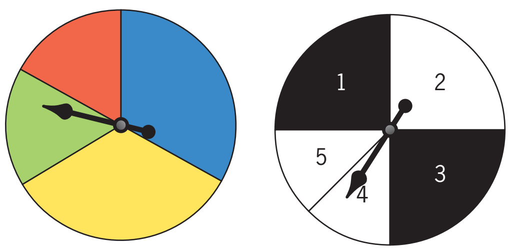
4. Two green tiles, one red tile, and a blue tile are put into a paper bag.

   a) What is the probability that a green tile is drawn, followed by a blue tile, assuming the first tile is replaced before the second tile is drawn?

   b) How does the answer to part a) change if the first tile drawn is not replaced?

   c) Explain why these answers are different.

5. **Thinking** Crazy Spinners is a game in which the two spinners below are spun at the same time.



   a) Player A wins a point if the result is Red-1. Player B wins a point if the result is Blue-4. Is this a fair game? Explain.

   b) **Open Question** Change the rules so that this game is almost but not quite fair.

6. Kevin works in car sales. Over a period of time he spoke with 400 customers. The experimental probability of a customer going for a test drive was 20%. If a customer went for a test drive, there was a 5% conditional probability that Kevin made a sale. Assuming that Kevin did not make a sale if there was no test drive, how many sales did Kevin make over this time period?

7. **Application** While playing an online adventure game, Briony finds herself lost in the Maze of Misfortune, as shown below:

   Briony is being pursued and has no time to second-guess any of her path decisions.

   a) Assuming she has no knowledge of the maze, what is the probability that Briony will successfully escape the Maze of Misfortune?

   b) What is the conditional probability that Briony will successfully navigate the maze given that she makes
   - her first path decision correctly?
   - her first two path decisions correctly?

8. Refer to Rolly's James Bond movie collection from page 37.

Actor	Number of Movies
Sean Connery	6
George Lazenby	1
Roger Moore	7
Timothy Dalton	2
Pierce Brosnan	4
Daniel Craig	2

   Suppose Rolly randomly picks a movie to watch, and then randomly picks a second movie without putting the first movie back on the shelf. Determine the probability of each of the following scenarios:

   a) Rolly will watch a Connery movie followed by a Moore movie.

   b) Rolly will watch two consecutive Dalton movies.

   c) Rolly will watch three consecutive Craig movies.

✓ Achievement Check

9. Petra wants to borrow her dad's car on Saturday, but so does her brother Alek. They decide to play Rock-Paper-Scissors to settle the dispute. Petra decides to go with Rock.

a) What is the probability that she will win the car on the first trial?

b) Assuming Petra continues with Rock, how likely is Petra to win the car if she and Alek play until a winner is declared?

c) Explain why these answers are different.

d) Is there any single choice that gives an advantage over any other? Explain using probability. What assumptions did you make?

Literacy Link

Rock-Paper-Scissors is a game in which two opponents each make one of the hand signals shown at exactly the same time. The winner is declared as follows:

- Paper covers Rock: Paper wins
- Scissors cuts Paper: Scissors wins
- Rock smashes Scissors: Rock wins

If both players make the same signal, the result is a draw and another trial is conducted.

10. Thinking Siko is a contestant on a TV game show called *Win a Million*. Each time she answers a multiple-choice question correctly, she wins money. If she picks a wrong answer, she is eliminated. If Siko does not know the right answer, she can use one of the following Helping Hands:

- *Quiz the Crowd*: She can poll the audience. The crowd has an experimental probability of being correct 85% of the time.
- *Double Up*: She can give two answers, instead of just one. If either is correct she stays in the game.
- *Rule One Out*: One of the incorrect answers is removed, leaving three choices.

Suppose Siko encounters three questions in a row to which she does not know the answers.

a) Assuming that she can use each Helping Hand only once during the game, and only once per question, what is the best estimated probability Siko has of staying alive through the three questions? What assumptions did you make.

b) How many more times is Siko likely to stay in the game if she uses all three Helping Hands than if she simply guesses at random on all three questions?

Extend

11. Thinking The Toronto Maple Leafs are facing the Montréal Canadiens in a best of seven playoff series. The first team to win four games wins the series. Ties are broken through sudden decision overtime. Assuming that the teams are evenly matched,

a) what are the odds in favour of either team sweeping the series, in which one team wins four consecutive games?

b) what are the odds against the series going a full seven games?

12. Open Question Refer to #11. How would your answers change if the teams were not very evenly matched? Pick one team as superior to the other to support your reasoning.

13. Suppose A and B are two dependent events. In general, will $P(A|B) = P(B|A)$? Use a bag of coloured tokens or an alternate scenario to illustrate your answer.

Processes

Reasoning and Proving
Can you use algebraic reasoning to prove that something is true? Can you use a counter-example to prove that something is not true?

14. Application In business, a common planning strategy is to use a decision tree.

a) Research this topic and write a brief report that addresses:

- What is a decision tree?
- What elements can it contain?
- How is it related to probability?
- Why is it a useful business strategy tool?

b) Include a real example of a decision tree and use it to support your answers to a).

Chapter 1 Review

Learning Goals

Section	After this section, I can
1.1	• use probability to describe the likelihood of something occurring • measure and calculate simple probabilities
1.2	• calculate theoretical probability
1.3	• recognize the difference between experimental probability and theoretical probability
1.4	• describe how an event can represent a set of probability outcomes • recognize how different events are related • calculate the probability of an event occurring
1.5	• describe and determine how the probability of one event occurring can affect the probability of another event occurring • solve probability problems involving multiple events

1.1 Simple Probabilities, pages 6–15

1. A mystery spinner is spun several times, producing the results shown in the table.

Colour	Count
Blue	24
Green	48
Yellow	51
Purple	26

a) Calculate the experimental probability of the spinner landing on each colour.

b) Sketch what this spinner could look like. Explain your reasoning.

c) Could the spinner look differently? Explain.

2. A quarterback successfully completed 21 of 35 pass attempts.

a) What is the experimental probability that the quarterback will complete a pass?

b) Suppose the quarterback throws 280 pass attempts over the course of a season. How many is he likely to complete, based on your answer to part a)?

3. Match each scenario with its most likely subjective probability. Justify your answers.

	Scenario	Subjective Probability, $P(A)$
a)	Canada will win at least one medal at the next Olympics.	0.1 0.25 0.9
b)	A person selected at random will be left-handed.	
c)	A randomly chosen high school student will be in grade 10.	

1.2 Theoretical Probability, pages 16–25

4. What is the theoretical probability of rolling each of the following sums with a pair of dice?

a) 2 **b)** 9

c) not 7 **d)** not a perfect square

5. A card is randomly drawn from a standard deck of cards. What is the theoretical probability that it will be

a) a club? **b)** an ace?

c) a face card?

6. A sports analyst predicts that a tennis player has a 25% chance of winning a tournament. What are the odds against winning?

1.3 Compare Experimental and Theoretical Probabilities, pages 26–33

7. A standard die is rolled 24 times and turns up a 3 six times.

a) What is the experimental probability of rolling a 3 on a given trial?

b) What is the theoretical probability of rolling a 3?

c) Explain why these answers are different.

8. Suppose two fair coins are flipped.

a) Draw a tree diagram to illustrate all possible outcomes.

b) Sketch a bar graph that shows the predicted relative frequency of each of the following events when a very large number of trials is carried out

- no heads
- one head
- two heads

c) Explain why your graph has the shape that it does.

1.4 Mutually Exclusive and Non-Mutually Exclusive Events, pages 34–43

9. A graphing calculator is programmed to randomly generate an integer value between 1 and 8. Determine the probability that the number will be

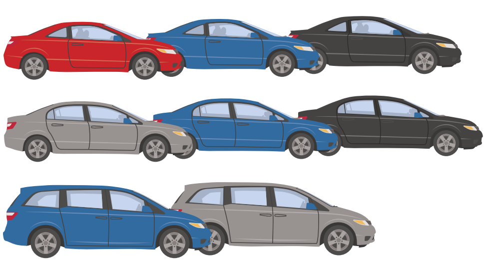
a) a five or an eight

b) a prime number or a perfect square

c) an even number or a seven

d) not a composite number or an odd number

10. A small vehicle rental company randomly assigns its vehicles to customers based on whatever happens to be available. The fleet is shown below.



Assume that each vehicle has an equal probability of being available at any given time. Determine the probability that a customer will randomly be assigned:

a) a coupe or a mini-van

b) a blue vehicle or a mini-van

c) a grey vehicle or a sedan

d) not a red vehicle or a coupe

Literacy Link

A *coupe* is a car with two passenger doors.
A *mini-van* is larger than a car but smaller than a van.
A *sedan* is a car with four passenger doors.

1.5 Independent and Dependent Events, pages 44–55

11. A standard die is rolled and a card is drawn from a standard deck of playing cards.

a) Which of the following is more likely to occur?

- an even value will be rolled and a heart will be drawn
- a composite value will be rolled and a face card will be drawn

b) Justify your answer with mathematical reasoning.

12. A bag has 3 red tiles, 1 yellow tile, and 2 green tiles.

a) What is the probability that a red tile is drawn, followed by a second red tile, if the first tile is replaced?

b) How does this value change if the first tile drawn is not replaced?

c) Explain why these answers are different.

13. Josiah has a 20% experimental probability of hitting the snooze button any morning when his alarm goes off. When he hits the snooze button, there is a 25% conditional probability that he misses his bus. He has never missed the bus when he has not hit the snooze button. If Josiah's alarm woke him 120 times over the course of the semester, how many times did Josiah miss his bus?

Chapter 1 Test Yourself

✓ Achievement Chart

Category	Knowledge/ Understanding	Thinking	Communication	Application
Questions	1, 2, 3	6, 10	5, 7	4, 8, 9

Multiple Choice

Choose the best answer for #1 to #3.

1. A married couple decides to have two children. Assuming that they do, what is the probability that they will either have two boys or two girls?

A 0.125 **B** 0.25

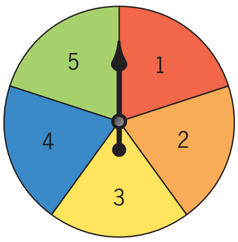
C 0.5 **D** $0.\overline{6}$

2. Natalie logged on to a social media website 50 times. Fifteen of those times she encountered a pop-up advertisement. What is the experimental probability that Natalie will see a pop-up at this site?

A 7.5% **B** 15%

C 30% **D** 70%

3. This spinner is spun 20 times and lands on green 5 times. Identify the true statement.



A The theoretical probability of landing on green is 20% and the experimental probability of landing on green is 20%.

B The theoretical probability of landing on green is 20% and the experimental probability of landing on green is 25%.

C The theoretical probability of landing on green is 25% and the experimental probability of landing on green is 20%.

D The theoretical probability of landing on green is 25% and the experimental probability of landing on green is 25%.

Short Answer

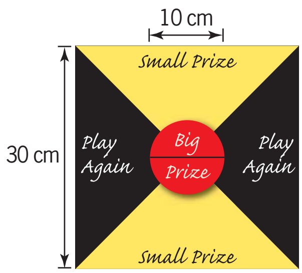
4. A fair coin is flipped four times. What is the probability that it will land heads exactly once?

5. Marlis feels 80% confident that she will pass her driver's exam.

a) What type of probability is Marlis using? Explain your choice.

b) What are the odds in favour of Marlis passing her driver's exam, based on her probability estimate? Justify your reasoning.

6. Tenzin is playing a carnival game in which he throws a dart at the target shown below. Assuming that he is equally likely to hit any point on the target, what is the probability Tenzin wins the following on a given throw?



a) a big prize

b) a small prize

7. In a game involving two standard dice, you win if you roll a sum of 7 or 11, or if you roll doubles (both dice showing the same number).

a) What are the odds against you winning this game?

b) Explain how you solved this problem.

8. Mr. Dobson's tie rack is shown below.

What is the probability that Mr. Dobson randomly selects

a) a solid blue tie or a polka dot tie?

b) a striped tie or a solid coloured tie?

c) a solid black tie or a striped tie?

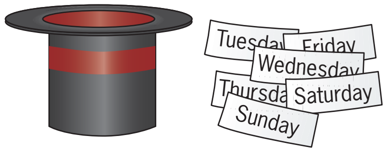
d) a solid coloured tie or a solid blue tie?

9. Bao has two pencils, a blue pen, and a red pen in his pencil case. Suppose he randomly withdraws one writing tool, followed by another, without replacing the first one. What is the probability that Bao will randomly draw

a) a red pen followed by a blue pen?

b) a pen followed by a pencil?

10. Abia is one of three servers who work at a restaurant that is open from Tuesday to Sunday. Every week the servers randomly draw slips of paper from a hat to decide which two days they will not have to work, in addition to Monday.



One week Abia gets to draw her two days first.

a) What is the probability that Abia will draw a weekend day (Saturday or Sunday) on her first draw?

b) What is the conditional probability that Abia will draw a second weekend day, given that her first draw was a weekend day?

c) What is the probability that Abia will get to enjoy a three-day weekend (Saturday to Monday)?

Chapter Problem

Game Analysis

At the beginning of this chapter you were asked to pick two or three games to analyse and report on the following questions:

1. What elements of the game involve strategy?

2. What elements of the game involve chance or probability?

3. What is the relative balance of strategy versus chance in this game?

Select at least three outcomes that are unique to the game, such as landing on a specific square, or rolling doubles twice in succession. Calculate the probability of each of these events occurring.

Present all of your findings for this project in one of the following formats:

- Written report
- Electronic slideshow
- Podcast
- Poster
- Other