

Experiment 1: Error and Data Analysis

Pre-lab Questions

[Give your answers on a separate piece of paper/s.]

1. (a) Distinguish among the types and causes of experimental errors.
(b) How can one reduce each type of experimental error?
2. Explain the difference between measurement precision and accuracy. How are these related to the types of error you have discussed in question (1)?
3. What determines the number of significant figures used in reporting measured values?
4. For a series of experimentally measured values give the definition of
 - (a) average or mean value
 - (b) the deviation from the mean
 - (c) average deviation
 - (d) standard deviation
5. Explain how one can express experimental result using the quantities discussed in (4).
6. (Optional) Explain how one can account for uncertainty in calculation of area from measurements of length and width of a rectangular block. [Let δl and δw be uncertainties in length and width measurement respectively. Express δA in terms of the two uncertainties]

1.1 Introduction

Our knowledge of the physical world around us is obtained from empirical observations and careful laboratory experiments in which we do measurements of physical quantities. It is important to understand how to express data collected from such measurements and how to analyze and draw meaningful conclusions from it. In this introductory study experiment we examine the type of experimental errors and some methods of error and data analysis that will be used in any experiment in which measurements are actually made.

1.2 Equipment Needed

Pencil/pen and ruler

Sheets of graph paper (optional)

Calculator

Or

A computer with some graphing software (excel will do the job well).

1.3 Theory

(i) Significant Figures

The significant figures of a (measured or calculated) quantity are the **meaningful** digits in it. There are conventions which you should learn and follow for how to express numbers so as to properly indicate their significant figures.

- Any digit that is not zero is significant. Thus 459 has three significant figures and 1.961 has four significant figures.
- Zeros between non zero digits are significant. Thus 2006 has four significant figures.
- Zeros to the left of the first non zero digit are not significant. Thus 0.000074 has only two significant figures. This is more easily seen if it is written in a scientific notation, i.e., 7.4×10^{-5} .
- For numbers with decimal points, zeros to the right of a non zero digit are significant. Thus 3.00 has three significant figures and 0.090 has two significant figures. For this reason it is important to keep the trailing zeros to indicate the actual number of significant figures.
- For numbers without decimal points, trailing zeros may or may not be significant. Thus, 400 indicates only one significant figure. To indicate that the trailing zeros are significant a decimal point must be added. For example, 400. has three significant figures, and 4×10^2 has one significant figure.
- Exact numbers have an infinite number of significant digits. For example, if there are two oranges on a table, then the number of oranges is 2.000... . Defined numbers are also like this. For example, the number of centimeters per inch (2.54) has an infinite number of significant digits, as does the speed of light (299792458 m/s).

There are also specific rules for how to consistently express the uncertainty associated with a number. In general, the last significant figure in any result should be of the same order of magnitude (i.e.. in the same decimal position) as the uncertainty. Also, the uncertainty should be rounded to one or two significant figures. Always work out the uncertainty after finding the number of significant figures for the actual measurement.

Example

$$9.82 \pm 0.02$$

$$10.0 \pm 1.5$$

$$4 \pm 1$$

The following table shows a few examples.

Wrong	Correct
9.81 ± 0.02671	9.81 ± 0.03
10.0 ± 2	10.0 ± 2.0
7 ± 0.2	7.0 ± 0.2

In practice, when doing mathematical calculations, it is a good idea to keep one more digit than is significant to reduce rounding errors. But in the end, the answer must be expressed with only the proper number of significant figures. After addition or subtraction, the result is significant only to the **place** determined by the largest last significant place in the original numbers. For example,

$$89.332 + 1.1 = 90.432$$

should be rounded to get 90.4 (the tenths place is the last significant place in 1.1). After multiplication or division, the **number** of significant figures in the result is determined by the original number with the smallest number of significant figures. For example,

$$(2.80)(4.5039) = 12.61092$$

should be rounded off to 12.6 (three significant figures like that of 2.80).

(ii) The Idea of Error

What is meant by "error"?

A measurement may be made of a quantity which has an accepted value which can be looked up in a handbook (e.g. the density of copper). The difference between the measurement and the accepted value is **not** what is meant by error. Such accepted values are not "right" answers. They are just measurements made by other people which have errors associated with them as well.

Nor does error mean "blunder". Reading a scale backwards, misunderstanding what we are doing or walloping into a partner's measuring apparatus are blunders which can be caught and should simply be disregarded.

Obviously, it cannot be determined exactly how far off a measurement is; if this could be done, it would be possible to just give a more accurate, corrected value.

Error, then, has to do with uncertainty in measurements that nothing can be done about. If a measurement is repeated, the values obtained will differ and none of the results can be preferred over the others. Although it is not possible to do anything about such error, it can be characterized. For instance, the repeated measurements may cluster tightly together or they may spread widely. This pattern can be analyzed systematically.

Classification of Error

Generally, errors can be divided into two broad and rough but useful classes: *systematic* and *random*.

Systematic errors are errors which tend to shift all measurements in a systematic way so their mean value is displaced. This may be due to such things as incorrect calibration of equipment, consistently improper use of equipment or failure to properly account for some effect. In a sense, a systematic error is rather like a blunder and large systematic errors can and must be eliminated in a good experiment. But small systematic errors will always be present. For instance, no instrument can ever be calibrated perfectly.

Other sources of systematic errors are external effects which can change the results of the experiment, but for which the corrections are not well known. In science, the reasons why several independent confirmations of experimental results are often required (especially using different techniques) is because different apparatus at different places may be affected by different systematic effects. Aside from making mistakes (such as thinking one is using the $\times 10$ scale, and actually using the $\times 100$ scale), the reason why experiments sometimes yield results which may be far outside the quoted errors is because of systematic effects which were not accounted for.

Random errors are errors which fluctuate from one measurement to the next. They yield results distributed about some mean value. They can occur for a variety of reasons.

- They may occur due to lack of sensitivity. For a sufficiently small change an instrument may not be able to respond to it or to indicate it or the observer may not be able to discern it.
- They may occur due to noise. There may be extraneous disturbances which cannot be taken into account.
- They may be due to imprecise definition.
- They may also occur due to statistical processes such as the roll of dice.

Random errors displace measurements in an arbitrary direction whereas systematic errors displace measurements in a single direction. Some systematic error can be substantially eliminated (or properly taken into account). Random errors are unavoidable and must be lived with.

Many times you will find results quoted with two errors. The first error quoted is usually the random error, and the second is called the systematic error. If only one error is quoted, then the errors from all sources are added together.

(iii) Expressing Experimental Error and Uncertainty

Percent Error

The purpose of some experiments is to determine the value of a well known physical quantity, for instance the value g-acceleration due to gravity.

The accepted or theoretical value of such a quantity is found in textbooks and physics handbooks is the most accurate value obtained through sophisticated experimental methods or mathematical methods.

The absolute difference between the experimental value X_E and the theoretical value X_T of a physical quantity X is given by the relation

Absolute difference = $|X_E - X_T|$. Note that absolute difference is a positive quantity.

The fractional error is given by

$$\begin{aligned}\text{Fractional error} &= \frac{\text{absolute difference}}{\text{theoretical value}} \\ &= \frac{|X_E - X_T|}{X_T}\end{aligned}\quad (1)$$

In most cases fractional difference is used to calculate percent error which is given by

$$\begin{aligned}\text{Percent error} &= \text{fractional error} \times 100\% \\ &= \frac{|X_E - X_T|}{X_T} \times 100\%\end{aligned}\quad (2)$$

Percent Difference

It is not always possible to find theoretical value for a physical quantity to be measured. In such circumstance we resort to comparison of results obtained from two equally dependable measurements. The comparison is expressed as a percent difference which is given by

$$\begin{aligned}\text{Percent difference} &= \frac{\text{absolute difference}}{\text{average}} \times 100\% \\ &= \frac{|X_1 - X_2|}{\left(\frac{X_1 + X_2}{2}\right)} \times 100\%,\end{aligned}\quad (3)$$

where X_1 and X_2 are results from the two methods.

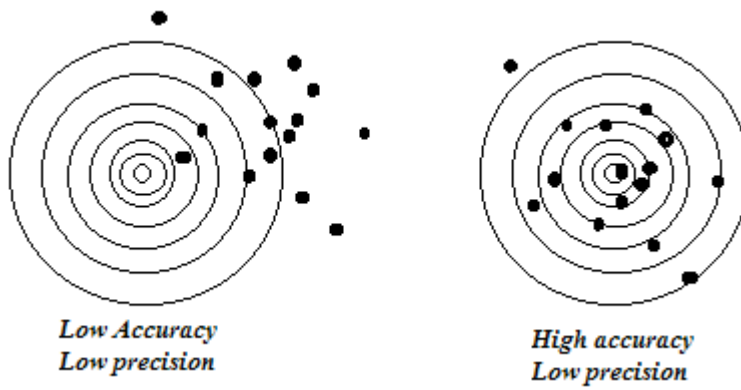
Statistical Tools

--Accuracy and precision

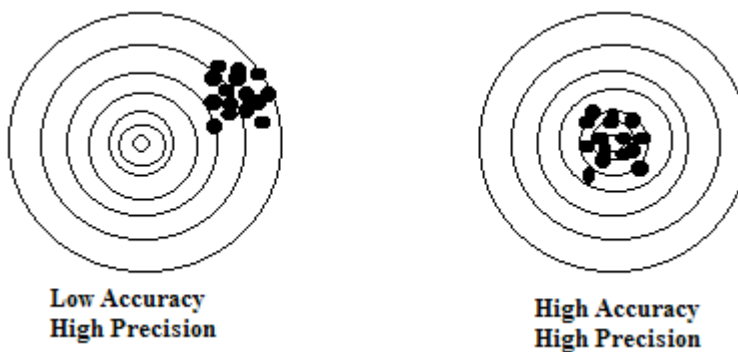
As discussed in above in most cases we do not know what the *real* value of the measured quantity is. If the measurement is repeated a number of times we might find that the result is a little different each time, however, not in a systematic or predictable way.

These variations in the result are due to *random* or *statistical errors*. Often the sources of random errors cannot be identified, and random errors cannot be predicted. Thus it is necessary to quantify random errors by means of statistical analysis. Simply by repeating an experiment or a measurement several times we will get an idea of how much the results vary from one measurement to the other. If there is little variation of the results we have a high *precision*, whereas large variations in the result indicate low *precision*. A way to visualize *accuracy* and *precision* is by the example of a dart board.

If we are a poor "dartist" our shots may be all over the board – and the wall; each shot hitting quite some distance from the other: both our accuracy (i.e. how close we are to the bull's eye) and our precision (i.e. the scatter of our shots) are low (below left).



A somewhat better "dartist" will at least consistently hit the board, but still with a wide scatter: now the accuracy of the throw is high, but the precision remains low (above right). Once the player gets consistent and there is not much scatter in the shots, the results may look like this:



Above left is not very accurate, however, the shots have high precision; once this is worked out systematically "drift to the right" the shots will be very accurate and very precise (above right).

What should be used to quantify the random error of a measurement? If only a single measurement X is made, we must decide how closely we can estimate the digit beyond those which can be read directly, perhaps 0.2 to 0.5 of the distance between the finest divisions on the measuring instrument.

i. *Mean Value*

Suppose an experiment were repeated many, say N , times to get, $X_1, X_2, \dots, X_N$ N measurements of the same quantity, X . The average or mean value, $\bar{X}$ of this measurement is given by

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_N}{N} = \frac{\sum_{k=1}^N X_k}{N} \quad (4)$$

ii. *Deviation from the mean*

Deviation from the mean or simply, deviation, is a quantitative description of how close the individual measurements are to each other and is given by

$$d_k = X_k - \bar{X}. \quad (5)$$

Note that the deviation can be positive or negative as some of the measured values are larger than the mean and others smaller. In ideal case the average of the deviations of

a set of measurements is zero. As a result, the mean of the deviations, $\frac{\sum_{k=1}^N d_k}{N}$, is not a useful quantity to describe the dispersion. To obtain the *average deviation*, or *mean deviation* of set of N measurements, the absolute value of individual deviations has to be calculated, i.e.,

$$\begin{aligned} \bar{d} &= \frac{|d_1| + |d_2| + |d_3| + \dots + |d_N|}{N} \\ &= \frac{\sum_{k=1}^N |d_k|}{N} \end{aligned} \quad (6)$$

This is a measure of the dispersion of our data about the mean (i.e., a measure of precession). The experimental value X_v of a measured quantity is given in the form

$$X_v = \bar{X} \pm \bar{d} \quad (7)$$

Note that the term $\pm$ gives a measure of the precision of the experimental value. The accuracy of the mean value experimental measurements should be expressed in terms of the percent error or percent difference.

iii. *Standard Deviation*

From statistical analysis we find that, for a data to have a Gaussian distribution means that the probability of obtaining the result X is,

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(X-X_0)^2}{2\sigma^2}} \quad (7)$$

where x_0 is most probable value and σ , which is called the standard deviation, determines the width of the distribution. Because of the law of large numbers this assumption will tend to be valid for random errors. And so it is common practice to quote error in terms of the standard deviation of a Gaussian distribution fit to the observed data distribution. This is the way we should quote error in our reports.

The mean is the most probable value of a Gaussian distribution. In terms of the mean, the **standard deviation** of any distribution is,

$$\sigma_x = \sqrt{\frac{\sum_{k=1}^N (X_k - \bar{X})^2}{N}} = \sqrt{\frac{\sum_{k=1}^N d_k^2}{N}} \quad (8)$$

The quantity σ^2 , the square of the standard deviation, is called the **variance**. The best estimate of the **true standard deviation** is,

$$\sigma_x = \sqrt{\frac{\sum_{k=1}^N (X_k - \bar{X})^2}{N-1}} \quad (9)$$

The reason why we divide by **N** to get the best estimate of the mean and only by **N-1** for the best estimate of the standard deviation needs to be explained. The true mean value of **x** is not being used to calculate the variance, but only the average of the measurements as the best estimate of it. Thus, $(X_k - \bar{X})^2$ as calculated is always a little bit smaller than $(X_k - X_{true})^2$, the quantity really wanted. In the theory of probability (that is, using the assumption that the data has a Gaussian distribution), it can be shown that this underestimate is corrected by using **N-1** instead of **N**.

Now we can express the experimental value X_v as

$$X_v = \bar{X} \pm \sigma_x \quad (10)$$

If one made one more measurement of **x** then (this is also a property of a Gaussian distribution) it would have some 68% probability of lying within $\bar{X} \pm \sigma_x$. Note that this means that about 30% of all experiments will disagree with the accepted value by more than one standard deviation!

However, we are also interested in the **error of the mean**, which is smaller than σ_x if there were several measurements. An exact calculation yields,

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} = \sqrt{\frac{\sum_{k=1}^N (X_k - \bar{X})^2}{N(N-1)}} \quad (11)$$

for the **standard error of the mean**. This means that, for example, if there were 20 measurements, the error on the mean itself would be = 4.47 times smaller than the error of each measurement. The number to report for this series of N measurements of x is $\bar{X} \pm \sigma_{\bar{x}}$ where $\frac{\sigma_x}{\sqrt{N}}$. The meaning of this is that if the N measurements of x were repeated there would be a 68% probability the new mean value of would lie within $\bar{X} \pm \sigma_{\bar{x}}$ (that is between $\bar{X} - \sigma_{\bar{x}}$ and $\bar{X} + \sigma_{\bar{x}}$). Note that this also means that there is a 32% probability that it will fall outside of this range. This means that out of 100 experiments of this type, on the average, 32 experiments will obtain a value which is outside the standard errors.

For a Gaussian distribution there is a 5% probability that the true value is outside of the range $\bar{X} \pm 2\sigma_{\bar{x}}$, i.e. twice the standard error, and only a 0.3% chance that it is outside the range of $\bar{X} \pm 3\sigma_{\bar{x}}$.

(iv) Propagation of Errors (optional)

Frequently, the result of an experiment will not be measured directly. Rather, it will be calculated from several measured physical quantities (each of which has a mean value and an error). What is the resulting error in the final result of such an experiment?

For instance, what is the error in $Z = A + B$ where A and B are two measured quantities with errors ΔA and ΔB respectively?

A first thought might be that the error in Z would be just the sum of the errors in A and B . After all,

$$(A + \Delta A) + (B + \Delta B) = (A + B) + (\Delta A + \Delta B) \quad (12)$$

and

$$(A - \Delta A) + (B - \Delta B) = (A + B) - (\Delta A + \Delta B). \quad (13)$$

But this assumes that, when combined, the errors in A and B have the same sign and maximum magnitude; that is that they always combine in the worst possible way. This could only happen if the errors in the two variables were perfectly correlated, (i.e.. if the two variables were not really independent).

If the variables are independent then sometimes the error in one variable will happen to cancel out some of the error in the other therefore, on the average, the error in $\mathbf{Z}$ will be less than the sum of the errors in its parts. A reasonable way to try to take this into account is to treat the perturbations in $\mathbf{Z}$ produced by perturbations in its parts as if they were "perpendicular" and added according to the Pythagorean theorem,

$$\Delta Z = \sqrt{\Delta A^2 + \Delta B^2} . \quad (14)$$

That is, if $\mathbf{A} = (100 \pm 3)$ and $\mathbf{B} = (6 \pm 4)$ then $\mathbf{Z} = (106 \pm 5)$ since $5 = \sqrt{3^2 + 4^2}$.

The derivation of the general rule for error propagation is beyond the scope general physics course. However, we give results for some common relationships (functional dependences) between measured quantities. Suppose there are two measurements, $\mathbf{A}$ and $\mathbf{B}$, and the final result is $\mathbf{Z} = \mathbf{F}(\mathbf{A}, \mathbf{B})$ for some function $\mathbf{F}$.

Relation between Z and (A,B)	Relation between errors ΔZ and $(\Delta A, \Delta B)$
1 $Z = A \pm B$	$(\Delta Z)^2 = (\Delta A)^2 + (\Delta B)^2$
2 $Z = AB \text{ or } A/B$	$\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$
3 $Z = A^n$	$\frac{\Delta Z}{Z} = n \frac{\Delta A}{A}$
4 $Z = \ln A$	$\Delta Z = \frac{\Delta A}{A}$
5 $Z = e^A$	$\Delta Z = \frac{\Delta A}{A}$

(v) Graphical Representation of Data

Experimental data is often presented in graphical form for reporting as well as obtain some information easily from properties of the graph.

Graphing procedures

In most cases quantities are plotted using Cartesian coordinate system in which the horizontal axis, referred to as abscissa, is labeled as x -axis and the vertical axis often called ordinate is labeled as y . The location of a point on the graph is specified by its coordinates x and y , given as (x, y) with respect to the origin, say O , the intersection of the axes.

When plotting data,

- choose axes scales that are easy to plot and read. The scale should not be too small to read.

- (b) If data points are plotted by hand, draw a smooth line connecting the points. If some plotting computer software is used choose plot with lines option.
- (c) Include the uncertainties in the experiment as error bar (mean deviation or standard deviation) on your graph.

In general graphs should have (see Figure 1)

- (i) Each axes labeled with the quantity plotted.
- (ii) The units of the quantities plotted.
- (iii) The title of the graph on the graph paper.

Why do we plot a data? The main reason for plotting a data in laboratory report is to explore the relationship between various measured quantities. Some quantities of interest in this case are

- (a) slope of the graph if the relationship is linear.
- (b) The degree of the polynomial or the coefficient of the highest degree polynomial for non linear relationship among measured quantities.

Note that these quantities can be determined directly from the graph with the use of proper least square fit (You may ask your instructor how to do a proper fit to your data).

Data Table 1		
t (s)	t ² (s ²)	y (m)
0.00	0.00	0.00
0.31	0.10	0.50
0.44	0.19	1.00
0.53	0.28	1.50
0.63	0.40	2.00
0.71	0.50	2.50
0.76	0.58	3.00
0.83	0.69	3.50
0.89	0.79	4.00
0.94	0.88	4.50
1.00	1.00	5.00

Free Fall, Height vs time graph

$$y = 4.8301t^2 + 0.2025t - 0.0093$$

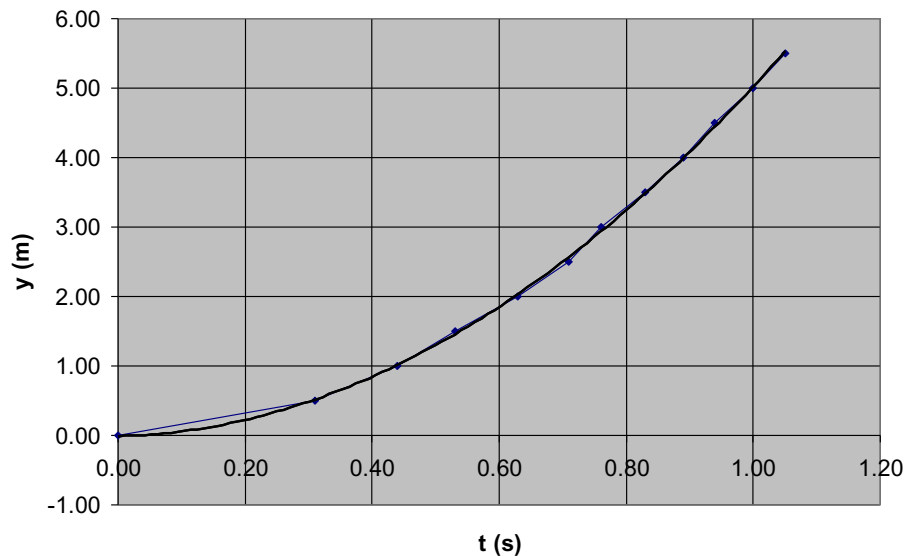


FIGURE 1

1.4 Procedure

Complete the exercise in the data and data analysis section. Show your calculations in detail and plot graphs as required.

1.5 Data and Data Analysis.

1. A metal cylinder is measured to have a length of 5.0 cm and a diameter of 1.26 cm. Compute the volume of the cylinder. Write your final answer with the correct number of significant figures.
2. In an experiment to determine the value of π , a sphere is measured to have an average value of 6.37 cm for its diameter and an average value of 20.11 cm for its circumference. What is the experimental value of π to the correct number of significant figures?
3. If the theoretical value of π is 3.142 what is the fractional error and percent error of the experimental value found in (2)?
4. In an experiment to measure the acceleration due to gravity g , two independent equally reliable measurements gave 9.67 m/s^2 and 9.88 m/s^2 . Determine
 - (i) the percent difference of the measurements
 - (ii) the percent error of their mean.[Take the theoretical value of g to be 9.81 m/s^2]
5. The equation of motion for an object in free fall starting from rest is

$$y = \frac{1}{2} g t^2,$$

where g is the acceleration due to gravity. The graphical form of this equation and the data plotted in Fig. 1 is called a parabola, a polynomial of degree 2, which has the general form $y = at^2 + bt + c$, where a , b , c are constants with values $a = g/2$, $b = 0$, $c = 0$ respectively for the present case.

For data given in table 1 plot y vs t^2 . This will be the plot of y vs t' where $t' = t^2$ which has the general form of $y = ax + b$, $a = g/2$ and $b = 0$.

- (a) Determine the experimental value of g from the slope of your graph.
- (b) Compute the percent error of the experimental value of g determined from the graph in (a)