

Problem 1 (6.5 pts). Find the general solution

$$[2x \sin(y) + 1] dx + [4x^2 \cos(y) + 3x \cot(y) + 5 \sin(2y)] dy = 0.$$

Hint. Use the integrating factor.

Solution. As $M = 2x \sin(y) + 1$, $N = 4x^2 \cos(y) + 3x \cot(y) + 10 \sin(y) \cos(y)$ we have

$$\begin{aligned} M_y - N_x &= 2x \cos(y) - [8x \cos(y) + 3 \cot(y)] = -6x \cos(y) - 3 \frac{\cos(y)}{\sin(y)} \implies \\ &\frac{N_x - M_y}{M} = 3 \cot(y). \end{aligned}$$

Therefore we are looking for an integrating factor $\mu(y)$, satisfying

$$\frac{\mu'}{\mu} = 3 \cot(y) \implies \ln(\mu) = 3 \int \cot(y) dy = 3 \ln(\sin(y))$$

(we take constant equal 0). Then $\mu = \sin^3(y)$.

After multiplication we get

$$\begin{aligned} [2x \sin^4(y) + \sin^3(y)] dx + \\ [4x^2 \sin^3(y) \cos(y) + 3x \sin^2(y) \cos(y) + 10 \sin^4(y) \cos(y)] dy = 0. \end{aligned}$$

Looking for $H(x, y)$ satisfying

$$H_x = 2x \sin^4(y) + \sin^3(y), \quad (1.1)$$

$$H_y = 4x^2 \sin^3(y) \cos(y) + 3x \sin^2(y) \cos(y) + 10 \sin^4(y) \cos(y). \quad (1.2)$$

(1.1) implies

$$H(x, y) = \int [2x \sin^4(y) + \sin^3(y)] dx = x^2 \sin^4(y) + x \sin^3(y) + h(y).$$

Then

$$H_y = 4x^2 \sin^3(y) \cos(y) + 3x \sin^2(y) \cos(y) + h'(y)$$

and comparing with (1.2) we see that

$$h'(y) = 10 \sin^4(y) \cos(y) \implies h(y) = \int 10 \sin^4(y) \cos(y) dy = 2 \sin^5(y);$$

we pick-up constant 0.

Finally

$$H(x, y) = x^2 \sin^4(y) + x \sin^3(y) + 2 \sin^5(y) = C$$

is a solution to the original problem.

□

Problem 2 (6.5 pts). Find the general solution by method of the undetermined coefficients:

$$y''' - 3y'' + 4y' + 2y = 20 \cosh(t) + 20 \cos(t);$$

Hint. All roots are integers (or complex integers).

Solution. Writing characteristic equation: $L(k) := k^3 - 3k^2 + 4k - 2 = 0$. Obviously, one root $k_1 = 1$; then $k_2 + k_3 = 2$, $k_1 k_2 = 2$ and they satisfy $k^2 - 2k + 2 = 0 \implies k_{1,2} = 1 \pm i$. Then

$$y^* = C_1 e^t + C_2 e^t \cos(t) + C_3 e^t \sin(t) \quad (2.1)$$

is a general solution to the homogeneous equation.

Solving inhomogeneous equations with RHE $f_1 = 10e^t$, $f_2 = 10e^{-t}$, $f_3 = \cos(t)$:

$$\begin{aligned} y_{p1} &= Ate^t, \\ y_{p2} &= Be^{-t}, \\ y_{p3} &= C \cos(t) + D \sin(t). \end{aligned}$$

Here $AL'(k)|_{k=1} = A(3k^2 - 6k + 4)|_{k=1} = 10 \implies A = 10$, $BL(-1) = -10B = 10 \implies B = -1$ and

$$\begin{aligned} (C + iD)L(i) &= (A + iB)(1 + 3i) = 20 \implies \\ C + iD &= \frac{20}{1 + 3i} = \frac{20(1 - 3i)}{10} = 2 - 6i \implies C = 2, D = -6. \end{aligned}$$

Then

$$\begin{aligned} y_{p1} &= 10te^t, \\ y_{p2} &= -e^{-t}, \\ y_{p3} &= 2 \cos(t) - 6 \sin(t). \end{aligned}$$

Finally

$$y = \underbrace{10te^t}_{y_{p1}} + \underbrace{-e^{-t}}_{y_{p2}} + \underbrace{2 \cos(t) - 6 \sin(t)}_{y_{p3}} + \underbrace{C_1 e^t + C_2 e^t \cos(t) + C_3 e^t \sin(t)}_{y^*}.$$

□

Problem 3 (7 pts). Find the general solution of

$$y''' - 2y'' - y' + 2y = \frac{12e^{2t}}{e^t + 1}.$$

Hint. All roots are integers (or complex integers).

Solution. Writing characteristic equation: $L(k) := k^3 - 2k^2 - k + 2 = 0$, with $L(k) = (k - 2)k^2 - (k - 2) = (k - 2)(k^2 - 1)$; then $k_1 = 1, k_2 = -1, k_3 = 2$. Then

$$y^* = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} \quad (3.1)$$

is a general solution to the homogeneous equation.

We are looking for solution to the inhomogeneous equation as (3.1) with unknown functions C_1, C_2, C_3 s.t.

$$\begin{cases} C_1' e^t + C_2' e^{-t} + C_3' e^{2t} = 0, \\ C_1' e^t - C_2' e^{-t} + 2C_3' e^{2t} = 0, \\ C_1' e^t + C_2' e^{-t} + 4C_3' e^{2t} = \frac{12e^{2t}}{e^t + 1}; \end{cases} \implies$$

$$\begin{cases} 2C_1' e^t + 3C_3' e^{2t} = 0, \\ C_1' e^t - C_2' e^{-t} + 2C_3' e^{2t} = 0, \\ 2C_1' e^t + 6C_3' e^{2t} = \frac{12e^{2t}}{e^t + 1}; \end{cases} \implies$$

$$C_3' e^{2t} = \frac{4e^{2t}}{e^t + 1}, \quad C_1' e^t = -\frac{6e^{2t}}{e^t + 1}, \quad C_2' e^{-t} = \frac{2e^{2t}}{e^t + 1} \implies$$

$$C_1 = -\int \frac{6e^t dt}{e^t + 1} = -6 \ln(e^t + 1) + c_1,$$

$$C_2 = \int \frac{2e^{3t} dt}{e^t + 1} = \int [2e^{2t} - 2e^t] dt + \int \frac{2e^t dt}{e^t + 1} = e^{2t} - 2e^t + 2 \ln(e^t + 1) + c_2,$$

$$C_3 = \int \frac{4 dt}{e^t + 1} = \int \frac{4e^{-t} dt}{1 + e^{-t}} = -4 \ln(1 + e^{-t}) + c_3 = -4 \ln(e^t + 1) + 4t + c_3.$$

Then

$$\begin{aligned} y = & \left[-6 \ln(e^t+1)+c_1\right] e^t + \left[e^{2t}-2e^t+2 \ln(e^t+1)+c_2\right] e^{-t} + \left[-4 \ln(e^t+1)+4t+c_3\right] e^{2t} = \\ & -6e^t \ln(e^t+1) + 2e^{-t} \ln(e^t+1) - 4e^{2t} \ln(e^t+1) + 4te^{2t} - 2 + \\ & c_1e^t + c_2e^{-t} + c_3e^{2t} \end{aligned}$$

with $c_1 := c_1 + 1$ in the last transition. □

Problem 4 (7 pts). Find the general solution $(x(t), y(t))$ of the system of ODEs

$$\begin{cases} x' = x - 2y + \sec(t) & -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ y' = x - y. \end{cases}$$

Hint. $\sec(t) = \frac{1}{\cos(t)}$.

Solution. Characteristic equation:

$$\begin{vmatrix} 1 - k & -2 \\ 1 & -1 - k \end{vmatrix} = k^2 + 1 = 0 \implies k_{1,2} = \pm i.$$

Finding eigenvector corresponding to $k_1 = i$

$$\begin{pmatrix} 1 - i & -2 \\ 1 & -1 - i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \alpha = (1 + i)\beta \implies \mathbf{e}_1 = \begin{pmatrix} 1 + i \\ 1 \end{pmatrix}$$

and $\mathbf{e}_2$ is complex conjugate. Then the general solution to the homogeneous equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \text{Re} \left[(C_1 + iC_2)(\cos(t) + i\sin(t)) \begin{pmatrix} 1 + i \\ 1 \end{pmatrix} \right] \implies$$

$$x = C_1(\cos(t) - \sin(t)) + C_2(-\cos(t) - \sin(t)) \quad (4.1)$$

$$y = C_1 \cos(t) - C_2 \sin(t). \quad (4.2)$$

We are looking for solution to inhomogeneous equation in the same form albeit with variable C_1, C_2 . Then

$$\begin{cases} C_1'(\cos(t) - \sin(t)) + C_2'(-\cos(t) - \sin(t)) = \sec(t), \\ C_1' \cos(t) - C_2' \sin(t) = 0 \end{cases} \implies$$

$$\begin{cases} C_1' \sin(t) + C_2' \cos(t) = -\sec(t), \\ C_1' \cos(t) - C_2' \sin(t) = 0 \end{cases} \implies$$

$$\begin{cases} C_1' = -\sec(t) \sin(t) = -\tan(t) \implies C_1 = \ln(\cos(t)) + c_1 \\ C_2' = -\sec(t) \cos(t) = -1 \implies C_2 = -t + c_2. \end{cases}$$

Finally

$$\begin{aligned}x &= [\ln(\cos(t)) + c_1](\cos(t) - \sin(t)) + [t - c_2](\cos(t) + \sin(t)) \\y &= [\ln(\cos(t)) + c_1] \cos(t) + [t - c_2] \sin(t) .\end{aligned}$$

□

Problem 5 (2+3+2+1=8 pts). For the system of ODEs

$$\begin{cases} x' = x(3x + 2y - 30), \\ y' = y(2y - x - 6). \end{cases}$$

- (a) Describe the locations of all critical points.
- (b) Classify their types (including whatever relevant: stability, orientation, etc.).
- (c) Sketch the phase portraits near the critical points.
- (d) Sketch the full phase portrait of this system of ODEs.

Solution. (a) Solving $x(3x + 2y - 30) = 0$, $y(2y - x - 6) = 0$ we get cases

$$\begin{array}{ll} x = 0, y = 0 & \implies A_1 = (0, 0), \\ x = 0, 2y - x - 6 = 0 & \implies A_2 = (0, 3) \\ y = 0, 3x + 2y - 30 = 0 & \implies A_3 = (10, 0), \\ 3x + 2y - 30 = 0, 2y - x - 6 = 0 & \implies A_4 = (6, 6). \end{array}$$

- (b) Linearizations at these points have matrices

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \\ \begin{pmatrix} -30 & 0 \\ 0 & -6 \end{pmatrix} & \begin{pmatrix} -24 & 0 \\ -3 & 6 \end{pmatrix} & \begin{pmatrix} 30 & 20 \\ 0 & -16 \end{pmatrix} & \begin{pmatrix} 18 & 12 \\ -6 & 12 \end{pmatrix} \end{array}$$

with eigenvalues

$$\{-30, -6\} \quad \{-24, 6\} \quad \{30, -16\} \quad \{15 - \sqrt{63}i, 15 + \sqrt{63}i\}$$

correspondingly.

Therefore A_1 is a stable node, A_2 and A_3 are saddles, A_4 is an unstable focal point and since left bottom number is $-6 < 0$ it is clockwise oriented.

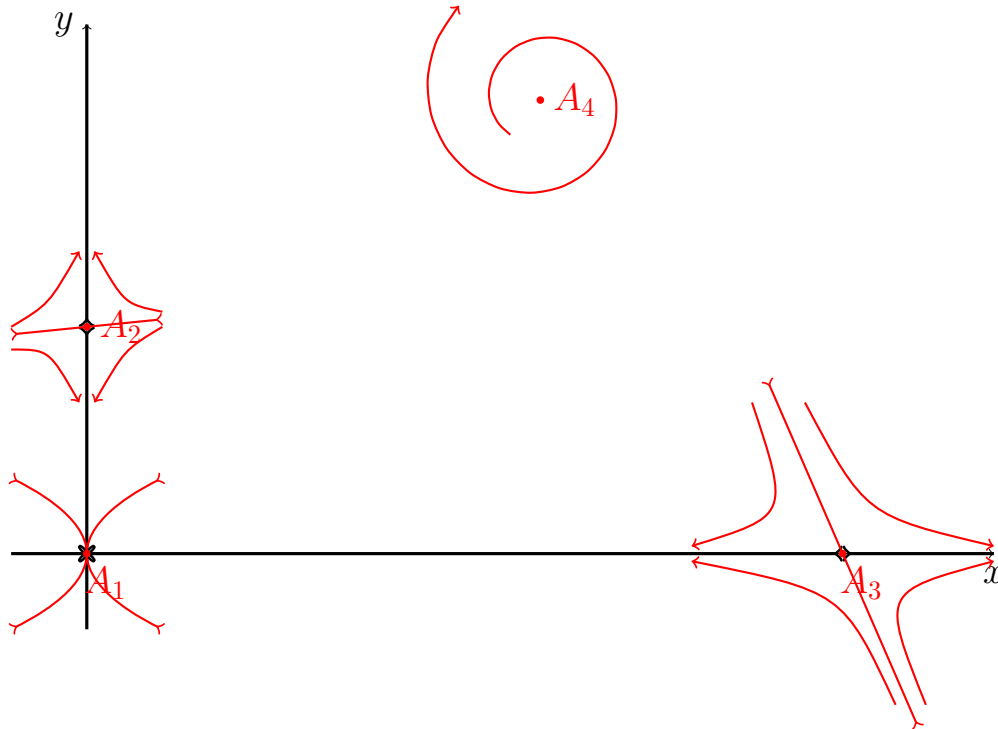
(c) Directions are

A_1 : $\mathbf{e}_1 = (1, 0)^T$, $\mathbf{e}_2 = (0, 1)^T$; Since $-30 = \lambda_1 < \lambda_2 = -6$, all trajectories have an entry directions $\pm \mathbf{f}_2$ (except two, which have entry directions $\pm \mathbf{f}_1$).

A_2 : $\mathbf{f}_1 = (10, 1)^T$ ($\lambda_1 = -24$)–stable, $\mathbf{f}_2 = (0, 1)^T$ ($\lambda_2 = 6$)–unstable.

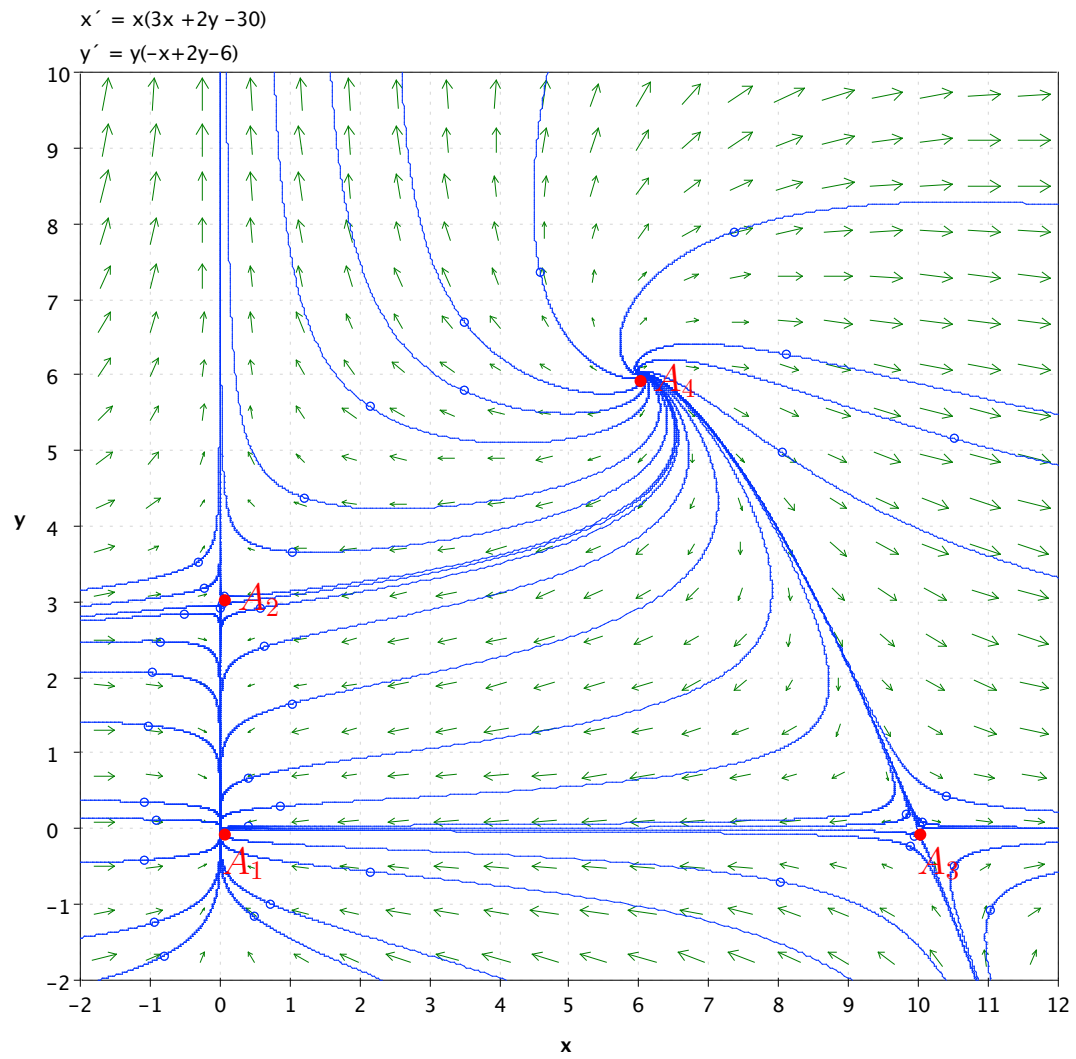
A_3 : $\mathbf{f}_1 = (1, 0)^T$ ($\lambda_1 = 30$)–unstable, $\mathbf{f}_2 = (1, -2.3)^T$ ($\lambda_2 = -16$)–stable.

(d) One should observe that either $x = 0$ in every point of the trajectory, or in no point; and that $y = 0$ in every point of the trajectory, or in no point. It allows us to make a “skeleton” of the phase portrait (thick black lines) on the figure; red lines are very approximate



Remark. A_4 may not look as a focal point on the computer-generated plot but it is. The reason for this disparity is that for one rotation (so angle increases by 2π) the radius increases in $\exp(2\pi 15/\sqrt{63}) \approx 130\,000$ times and this is a really big number.

□



Continue

Problem 6 (.5+1.5+2+1=5 pts). For the system of ODEs

$$\begin{cases} x' = 2y(x^2 + y^2 + 4), \\ y' = -2x(x^2 + y^2 - 16) \end{cases}$$

- (a) Find stationary points.
- (b) Linearize the system at stationary points and sketch the phase portrait of this linear system.
- (c) Find the equation of the form $H(x, y) = C$, satisfied by the trajectories of the nonlinear system.
- (d) Sketch the full phase portrait.

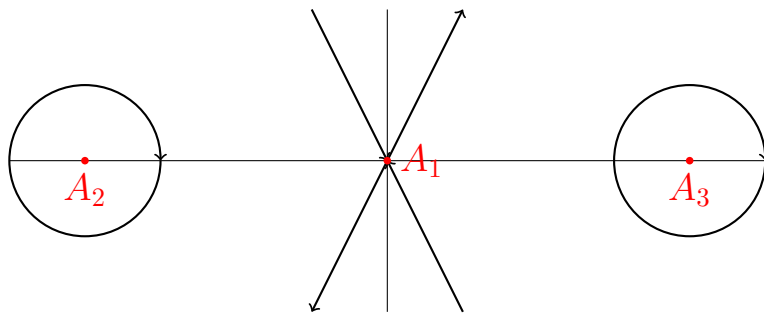
Solution. (a) Since $x^2 + y^2 + 4 > 0$ all stationary points are at $y = 0$, and either $x = 0$ or $x^2 + y^2 - 16 = 0$; thus we have $A_1 = (0, 0)$ and $A_{2,3} = (\pm 4, 0)$.

(b) Let $f = 2y(x^2 + y^2 + 4)$, $g = -2x(x^2 + y^2 - 16)$.

At A_1 we have $f_x = 0$, $f_y = 8$, $g_x = 32$, $g_y = 0$ and linearization is

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} 0 & 8 \\ 32 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

with eigenvalues ± 16 and eigenvectors $\begin{pmatrix} 1 \\ \pm 2 \end{pmatrix}$ so it is a saddle: At points $A_{2,3}$ we



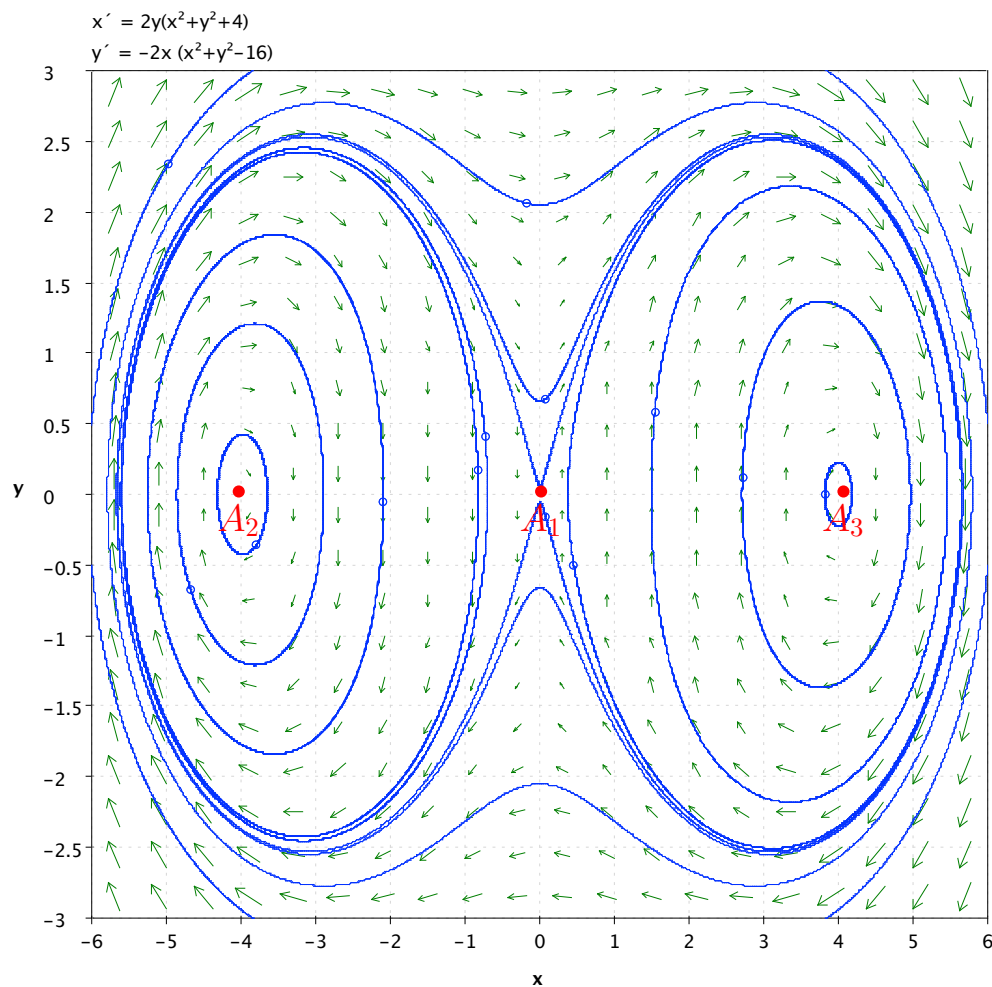
have $f_x = 0$, $f_y = 40$, $g_x = -64$, $g_y = 0$ with eigenvalues $16\sqrt{5}i$ and $-16\sqrt{5}i$; in virtue of (c) it is a center, and since the bottom left element is negative it is clockwise.

(c) Rewriting the sistem as

$$2x(x^2 + y^2 - 16) dx + 2y(x^2 + y^2 + 4) dy = 0$$

one can check that the form on the left is exact; $H_x = 2(x^3 + xy^2 - 16x) \implies H(x, y) = \frac{1}{2}(x^4 + 2x^2y^2 - 32x^2) + h(y)$ and plugging to $H_y = 2x^2y + h'(y) = 2yx^2 + 2y^3 + 8y \implies h'(y) = 2y^3 + 8y \implies h(y) = \frac{1}{2}y^4 + 4y^2$ (we take a constant equal 0). Then

$$H(x, y) = \frac{1}{2}(x^4 + 2x^2y^2 + y^4 - 32x^2 + 8y^2).$$



□