

Solutions

Problem 1 (7.5 pts). Find the general solution

$$(3x + 2y) dx + \left(x + \frac{6y^2}{x}\right) dy = 0.$$

Hint. Use the integrating factor.

Solution. As $M = 3x + 2y$, $N = x + \frac{6y^2}{x}$ we have

$$M_y - N_x = 2 - \left[1 - \frac{6y^2}{x^2}\right] = 1 + \frac{6y^2}{x^2} \implies \frac{N_x - M_y}{N} = \frac{1}{x}.$$

Therefore we are looking for an integrating factor $\mu(x)$, satisfying

$$\frac{\mu'}{\mu} = \frac{1}{x} \implies \ln(\mu) = \int \frac{1}{x} dx = \ln(x)$$

(we take constant equal 0). Then $\mu = x$.

After multiplication we get

$$(3x + 2xy) dx + (x^2 + 6y^2) dy = 0.$$

Looking for $H(x, y)$ satisfying

$$H_x = 3x^2 + 2xy, \tag{1.1}$$

$$H_y = x^2 + 6y^2. \tag{1.2}$$

(??) implies

$$H(x, y) = \int (3x^2 + 2xy) dx = x^3 + x^2y + h(y).$$

Then

$$H_y = x^2 + h'(y)$$

and comparing with (??) we see that

$$h'(y) = 6y^2 \implies h(y) = 2y^3;$$

we pick-up constant 0.

Finally

$$H(x, y) = x^3 + x^2y + 2y^3 = C$$

is a solution to the original problem. □

Problem 2 (7.5 pts). Find the general solution by method of the undetermined coefficients:

$$y''' - y'' + 2y = 40 \cosh(t) + 80 \cos(t);$$

Hint. All roots are integers (or complex integers).

Solution. Writing characteristic equation: $L(k) := k^3 - k^2 + 2 = 0$. Obviously, one root $k_1 = -1$; then $k_2 + k_3 = 2$, $k_1 k_2 = 2$ and they satisfy $k^2 - 2k + 2 = 0 \implies k_{1,2} = 1 \pm i$. Then

$$y^* = C_1 e^{-t} + C_2 e^t \cos(t) + C_3 e^t \sin(t) \quad (2.1)$$

is a general solution to the homogeneous equation.

Solving inhomogeneous equations with RHE $f_1 = 20e^t$, $f_2 = 20e^{-t}$, $f_3 = 80 \cos(t)$:

$$\begin{aligned} y_{p1} &= A e^t, \\ y_{p2} &= B t e^{-t}, \\ y_{p3} &= C \cos(t) + D \sin(t) = \operatorname{Re}((C - iD)e^{it}). \end{aligned}$$

Here $AL(1) = 2A = 20 \implies A = 10$, $BL'(-1) = B(3k^2 - 2k)|_{k=-1} = 5B = 20 \implies B = 4$ and

$$\begin{aligned} (C - iD)L(i) &= (C - iD)(3 - i) = 80 \\ C - iD &= \frac{80}{3 - i} = 80(3 + i)/10 = 24 + 8i \implies C = 24, D = -8 \end{aligned}$$

Then

$$\begin{aligned} y_{p1} &= 10e^{-t}, \\ y_{p2} &= 4te^{-t}, \\ y_{p3} &= 24 \cos(t) - 8 \sin(t). \end{aligned}$$

Finally

$$y = \underbrace{10e^{-t}}_{y_{p1}} + \underbrace{4te^{-t}}_{y_{p2}} + \underbrace{24 \cos(t) - 8 \sin(t)}_{y_{p3}} + \underbrace{C_1 e^{-t} + C_2 e^t \cos(t) + C_3 e^t \sin(t)}_{y^*}.$$

□

Problem 3 (7.5 pts). Find the general solution of

$$y''' + 2y'' - y' - 2y = \frac{12e^t}{e^{2t} + 1}.$$

Hint. Use method of variations. All roots are integers (or complex integers).

Solution. Characteristic equation is $L(k) = k^3 + 2k^2 - k - 2 = (k+2)(k^2 - 1) = (k-1)(k+1)(k+2)$ and roots are $k_1 = 1, k_2 = -1, k_3 = -2$ and

$$y = C_1 e^t + C_2 e^{-t} + C_3 e^{-2t} \quad (3.1)$$

is a solution to homogeneous equation. We look for solution to our equation in the same form, albeit with variable C_1, C_2, C_3 . Then

$$\begin{cases} C_1' e^t + C_2' e^{-t} + C_3' e^{-2t} = 0, \\ C_1' e^t - C_2' e^{-t} - 2C_3' e^{-2t} = 0, \\ C_1' e^t + C_2' e^{-t} + 4C_3' e^{-2t} = \frac{12e^t}{e^{2t} + 1}. \end{cases}$$

Adding and subtracting the first and the second equations, we see that $C_1' e^t = \frac{1}{2} C_3' e^{-2t}$ and $C_2' e^{-t} = -\frac{3}{2} C_3' e^{-2t}$ and then

$$C_3' e^{-2t} = \frac{4e^t}{e^{2t} + 1}, \quad C_1' e^t = \frac{2e^t}{e^{2t} + 1}, \quad C_2' e^{-t} = -\frac{6e^t}{e^{2t} + 1}.$$

and

$$C_1 = \int \frac{2dt}{e^{2t} + 1} = \int \left(2 - \frac{2e^{2t}}{e^{2t} + 1} \right) dt = 2t - \ln(e^{2t} + 1) + c_1,$$

$$C_2 = - \int \frac{6e^{2t} dt}{e^{2t} + 1} = -3 \ln(e^{2t} + 1) + c_2,$$

$$C_3 = \int \frac{4e^{3t} dt}{e^{2t} + 1} = 4 \int \left(e^t - \frac{e^t}{e^{2t} + 1} \right) dt = 4e^t - 4 \arctan(e^t) + c_3$$

and plugging into (3.1) we get

$$y = \left(2t - \ln(e^{2t} + 1) + c_1 \right) e^t + \left(-3 \ln(e^{2t} + 1) + c_2 \right) e^{-t} + \left(4e^t - 4 \arctan(e^t) + c_3 \right) e^{-2t}.$$

□

Problem 4 (7.5 pts). Find the general solution $(x(t), y(t))$ of the system of ODEs

$$\begin{cases} x'_t = -2y + \tan(t) , \\ y'_t = 2x . \end{cases}$$

Solution. Characteristic equation:

$$\begin{vmatrix} -k & -2 \\ 2 & -k \end{vmatrix} = k^2 + 4 = 0 \implies k = \pm 2i.$$

Finding eigenvector corresponding to $k_1 = 2i$

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \implies \beta = -i\alpha \implies \mathbf{e}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and $\mathbf{e}_2$ is complex-conjugate: $\mathbf{e}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Then the general solution to the homogeneous equation is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \operatorname{Re}(C_1 + iC_2) \underbrace{(\cos(2t) + i \sin(2t))}_{=e^{2it}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \begin{pmatrix} C_1 \cos(2t) - C_2 \sin(2t) \\ C_1 \sin(2t) + C_2 \cos(2t) \end{pmatrix}. \quad (4.1) \end{aligned}$$

We are looking for solution to inhomogeneous equation in the same form albeit with variable C_1, C_2 . Then

$$\begin{aligned} \begin{cases} C'_1 \cos(2t) - C'_2 \sin(2t) = \tan(t), \\ C'_1 \sin(2t) + C'_2 \cos(2t) = 0 \end{cases} &\implies \\ \begin{cases} C'_1 = \cos(2t) \tan(t) = (2 \cos^2(t) - 1) \tan(t) = \sin(2t) - \tan(t), \\ C'_2 = \sin(2t) \tan(t) = 2 \sin^2(t) = 1 - \cos(2t) \end{cases} &\implies \\ \begin{cases} C_1 = -\frac{1}{2} \cos(2t) + \ln(\cos(t)) + c_1, \\ C_2 = t - \frac{1}{2} \sin(2t) + c_2 \end{cases} & \end{aligned}$$

and finally

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} [-\frac{1}{2} \cos(2t) + \ln(\cos(t)) + c_1] \cos(2t) - [t - \frac{1}{2} \sin(2t) + c_2] \sin(2t) \\ [-\frac{1}{2} \cos(2t) + \ln(\cos(t)) + c_1] \sin(2t) + [t - \frac{1}{2} \sin(2t) + c_2] \cos(2t) \end{pmatrix}.$$

□

Problem 5 (9 pts). For the system of ODEs

$$\begin{cases} x' = x(10 - 2x - 3y), \\ y' = y(3x + 2y - 12) \end{cases}$$

- (a) Describe the locations of all critical points.
- (b) Classify their types (including whatever relevant: stability, orientation, etc.).
- (c) Sketch the phase portraits near the critical points.
- (d) Sketch the full phase portrait of this system of ODEs.

Solution. (a) Solving $x(10 - 2x - 3y) = 0$, $y(3x + 2y - 12) = 0$ we get cases

$$\begin{aligned} x = 0, y = 0 & \implies A_1 = (0, 0), \\ x = 0, 3x + 2y - 12 = 0 & \implies A_2 = (0, 6) \\ y = 0, 10 - 2x - 3y = 0 & \implies A_3 = (5, 0), \\ 10 - 2x - 3y = 0, 3x + 2y - 12 = 0 & \implies A_4 = \left(\frac{16}{5}, \frac{6}{5}\right). \end{aligned}$$

- (b) Linearizations at these points have matrices

$$\begin{aligned} A_1 = (0, 0) \quad A_2 = (0, 6) \quad A_3 = (5, 0) \quad A_4 = \left(\frac{16}{5}, \frac{6}{5}\right) \\ \begin{pmatrix} 10 & 0 \\ 0 & -12 \end{pmatrix} \quad \begin{pmatrix} -8 & 0 \\ 18 & 12 \end{pmatrix} \quad \begin{pmatrix} -10 & -15 \\ 0 & 3 \end{pmatrix} \quad \begin{pmatrix} -\frac{32}{5} & -\frac{48}{5} \\ \frac{18}{5} & \frac{12}{5} \end{pmatrix} \end{aligned}$$

with eigenvalues

$$\{10, -12\} \quad \{-8, 12\} \quad \{-10, 3\} \quad \{-2 \pm \sqrt{\frac{76}{5}}i\}$$

correspondingly; the last one is from characteristic equation

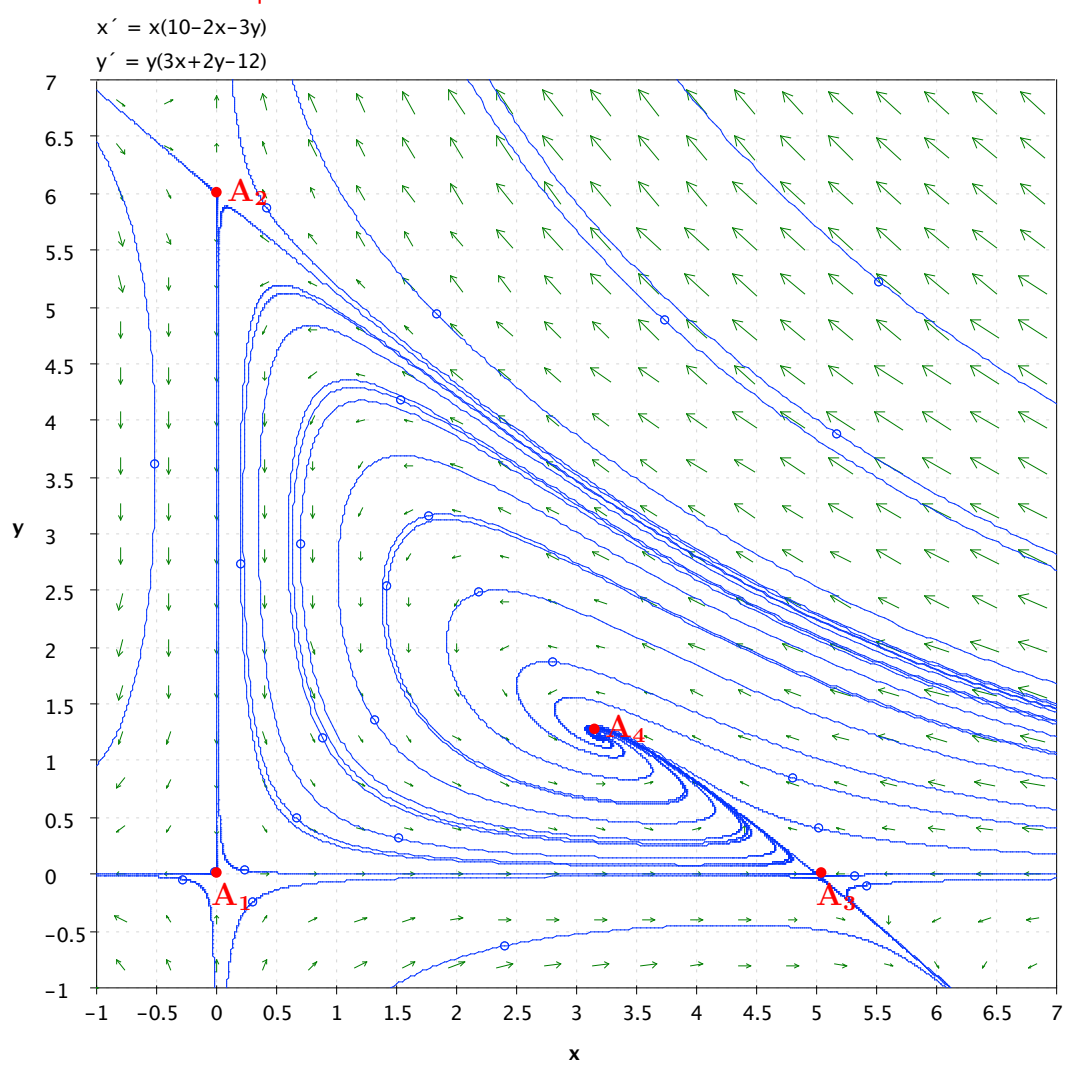
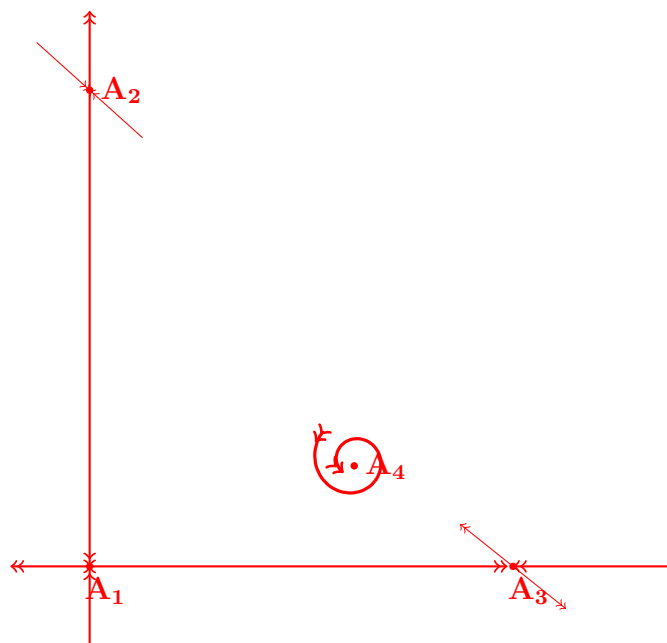
$$k^2 + 4k + \frac{96}{5} = 0.$$

Therefore A_1 , A_2 and A_3 are saddles, A_4 is a focal stable point and since the bottom-left number is positive, it is counter-clockwise oriented. Directions are

$$\begin{aligned} A_1: \mathbf{f}_1 = (1, 0)^T \text{--unstable, } \mathbf{f}_2 = (0, 1)^T \text{--stable } (k_1 > 0 > k_2); \\ A_2: \mathbf{f}_1 = (1, -\frac{9}{10})^T \text{--stable, } \mathbf{f}_2 = (0, 1)^T \text{--unstable } (k_1 < 0 < k_2). \\ A_3: \mathbf{f}_1 = (1, 0)^T \text{--stable, } \mathbf{f}_2 = (1, -\frac{13}{15})^T (k_1 < 0 < k_2) \text{--suntable.} \end{aligned}$$

One should observe that either $x = 0$ in every point of the trajectory, or in no point; and that $y = 0$ in every point of the trajectory, or in no point. It allows us to make a “skeleton” of the phase portrait.

□



Problem 6 (6 pts). For system of ODEs

$$\begin{cases} x' = x^2y - 3y^2 + 2x, \\ y' = -xy^2 + 3x^2 - 2y. \end{cases}$$

- (a) Find stationary points.
- (b) Linearize the system at stationary points and sketch the phase portrait of this linear system.
- (c) Find the equation of the form $H(x, y) = C$, satisfied by the trajectories of the nonlinear system.
- (d) Sketch the full phase portrait.

Solution. (a) Solving $x^2y - 3y^2 + 2x = 0$, $-xy^2 + 3x^2 - 2y = 0$ we multiply the these equations by y , x respectively and add: $3x^3 - 3y^3 = 0 \implies x = y$. Then $x^3 - 3x^2 + 2 = 0$ results in $x = 0$, $x = 1$, $x = 2$ and $A_1(0, 0)$, $A_2(1, 1)$, $A_3(2, 2)$.

- (b) Linearizations at these points have matrices

$$\begin{array}{ccc} A_1 = (0, 0) & A_2 = (1, 1) & A_3 = (2, 2) \\ \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} & \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} & \begin{pmatrix} 10 & -8 \\ 8 & -10 \end{pmatrix} \end{array}$$

with eigenvalues

$$\{2, -2\} \qquad \{-\sqrt{41}i, \sqrt{41}i\} \qquad \{-6, 6\}$$

correspondingly.

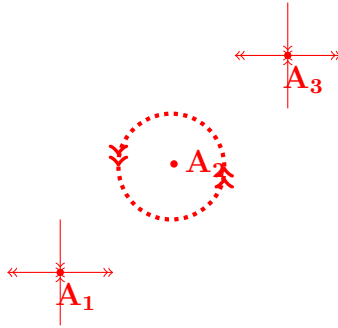
Therefore A_1 and A_3 are saddles, A_2 is either center or focus and since bottom left $5 > 0$ it is counter-clockwise oriented.

- (c) Directions are

A_1 : $\mathbf{f}_1 = (1, 0)^T$ —unstable direction, $\mathbf{f}_2 = (0, 1)^T$ —stable direction (since $k_1 > 0 > k_2$).

A_3 : $\mathbf{f}_1 = (1, -2)^T$ —stable direction, $\mathbf{f}_2 = (2, 1)^T$ —unstable.

- (d) One should observe that either $x = 0$ in every point of the trajectory, or in no point; and that $y = 0$ in every point of the trajectory, or in no point. It allows us to make a “skeleton” of the phase portrait (thick black lines) on the figure; red lines are very approximate.



(e) Rewriting equation as

$$\begin{aligned} (-xy^2 + 3x^2 - 2y) dx - (x^2y - 3y^2 + 2x) dy &= 0 \iff \\ (-xy^2 + 3x^2 - 2y) dx + (-x^2y + 3y^2 - 2x) dy & \end{aligned}$$

one can check that it is exact. Then

$$\begin{aligned} H_x = -xy^2 + 3x^2 - 2y &\implies H(x, y) = -\frac{1}{2}x^2y^2 + x^3 - 2xy + h(y) \implies \\ H_y = -x^2y - 2x + h'(y) &\stackrel{?}{=} -x^2y + 3y^2 - 2x \implies h'(y) = 3y^2 \implies h(y) = y^3 \end{aligned}$$

(we do not need the constant here) and solution is

$$H(x, y) = -\frac{1}{2}x^2y^2 + x^3 + y^3 - 2xy = C$$

Therefore $(1, 1)$ must be center. One can check easily that it is a local minimum of $H(x, y)$ but we do not need it. $\square$

