

FINA 306 Financial Economics

Assignment

Due at 3:40pm on Tuesday, October 9, 2018

Explain your working in detail, and please don't leave starting this assignment until the last minute. Most of the questions contain twists that make them more difficult than they appear on a first reading.

1. **(Decision-making under uncertainty)** Suppose that there are two states of nature that can occur at date 1. In state 1 the All Blacks win the Rugby World Cup in 2019, in state 2 they do not win the Rugby World Cup. Suppose that the All Blacks will win with probability p satisfying $0 < p < 1$.

Your utility equals

$$u_1(c) = \log(\alpha + c)$$

in state 1 and

$$u_2(c) = \log(c)$$

in state 2, where α is a positive constant and c is your level of consumption in that state, and you maximize the expected value of your utility. Suppose that you will have w to spend on consumption at the time of the Rugby World Cup with certainty.

Consider the following asset: for each unit of the asset you hold, you must pay one unit of consumption if the All Blacks win but receive q units of consumption if the All Blacks lose.

- (a) For what value of q is the expected payoff from each unit of the asset equal to 0?
- (b) How many units of the asset should you hold, assuming that the asset is free at date 0? Compare your answer in the general case to your answer in the special case where the asset's expected payoff is zero.

Hint: Let x denote the number of units of the asset that you hold, derive an expression for your expected utility as a function of x , and maximize.

- (c) Use this model to discuss the merits of the suggestion that All Blacks supporters should bet against the team. Should more passionate supporters make bigger or smaller bets?
2. **(The Arrow-Debreu economy)** There is a single consumption good, two possible states of nature, and two types of expected-utility maximizing investors, made up of n_A "Type A" and n_B "Type B" investors. Each investor, regardless of type, is endowed with

one unit of the consumption good in state 1 and two units in state 2. Type A investors have von Neumann-Morgenstern utility function $u^A(c) = 1 - e^{-\alpha_A c}$ and type B investors have von Neumann-Morgenstern utility function $u^B(c) = 1 - e^{-\alpha_B c}$, where α_A and α_B are positive constants. All investors believe that the two states are equally likely to occur. State prices are θ_1 and θ_2 .

- (a) Derive an expression for the aggregate demand for the Arrow-Debreu security paying out in state 1.

For parts (b) and (c), assume that $n_A = 40$, $n_B = 60$, $\alpha_A = 0.1$, and $\alpha_B = 2$.

- (b) Plot aggregate demand for the Arrow-Debreu security paying out in state 1 as a function of the relative price θ_1/θ_2 . You might want to plot the function using EXCEL or something similar. Remember to use `=LN(X)` in EXCEL, not `=LOG(X)`.
- (c) Calculate the market-clearing state prices. It is possible to calculate an exact solution as a function of $(n_A, n_B, \alpha_A, \alpha_B)$, but it is acceptable to solve the market-clearing condition using a function such as `GoalSeek` or `Solver` in EXCEL. Calculate how trade in Arrow-Debreu securities (re)allocates risk in this special case. Calculate the risk-neutral probabilities.
- (d) Explain what this example shows about the role that financial markets can play in allowing agents to allocate risk.

For part (e), continue to assume that $n_A = 40$, $n_B = 60$, and $\alpha_A = 0.1$, but now α_B takes various values.

- (e) Complete the following table and use the results to explain the effect of risk attitudes on equilibrium state prices and the allocation of risk in equilibrium. When constructing the table, vary α_B and keep all other parameters equal to their values above.

α_B	θ_1^*/θ_2^*	$\hat{c}_1^A$	$\hat{c}_2^A$	$\hat{c}_1^B$	$\hat{c}_2^B$	π_1^*	π_2^*
0.1							
0.25							
0.5							
1							
2							
4							
8							
16							

3. **(Implications for asset prices)** In this question we consider the special case of the asset-pricing model derived in Topic 4 where the stochastic discount factor is a simple function of the returns on the market portfolio. Take as the starting point the asset-pricing formula

$$V = E[\tilde{m}\tilde{z}], \quad (1)$$

where V is the date 0 price of an arbitrary asset, $\tilde{z}$ is its date 1 payoff, and $\tilde{m}$ is the stochastic discount factor. Let $\tilde{r} = \tilde{z}/V - 1$ denote the rate of return on the asset.

- (a) Use the definition of $\tilde{r}$ and equation (1) to show that the rate of return satisfies the following equation:

$$E[\tilde{m}(1 + \tilde{r})] = 1. \quad (2)$$

For the remainder of this question, suppose that the stochastic discount factor equals $\tilde{m} = a - b\tilde{r}_m$, for some constants a and b , where $\tilde{r}_m$ is the rate of return on the market portfolio.

- (b) Equation (2) must hold when $\tilde{r}$ is the rate of return on the risk-free asset. Use this property to show that a and b must satisfy

$$1 = (a - b\bar{r}_m)(1 + r_f), \quad (3)$$

where r_f is the risk-free interest rate and $\bar{r}_m$ is the expected rate of return on the market portfolio.

- (c) Equation (2) must also hold when $\tilde{r}$ is the rate of return on the market portfolio. Use this property to show that a and b must also satisfy

$$1 = a(1 + \bar{r}_m) - b(\bar{r}_m + \sigma_m^2 + \bar{r}_m^2), \quad (4)$$

where σ_m^2 is the variance of the rate of return on the market portfolio.

- (d) Solve equations (3) and (4) for a and b and use your answer to show that the stochastic discount factor equals

$$\tilde{m} = \frac{1}{1 + r_f} \left(1 - \lambda_m \cdot \frac{\tilde{r}_m - \bar{r}_m}{\sigma_m} \right),$$

where

$$\lambda_m = \frac{\bar{r}_m - r_f}{\sigma_m}.$$

What does λ_m represent? What is $(\tilde{r}_m - \bar{r}_m)/\sigma_m$?

- (e) Rearrange equation (2) to show that the expected rate of return for an arbitrary asset equals

$$E[\tilde{r}] = r_f + \frac{\text{Cov}[r_m, \tilde{r}]}{\text{Var}[r_m]} (\bar{r}_m - r_f).$$

- (f) Discuss what you have found.

4. **(The asset-market economy)** Suppose there are two assets available to investors: a risk-free asset with a price of $p_1 = 1$ and that pays R_f with certainty at date 1, and a risky asset with a price of $p_2 = 1$ and that pays $\tilde{R}$ at date 1, where $\tilde{R}$ is normally distributed with mean $\bar{R}$ and variance σ^2 . An expected utility maximizer with von Neumann-Morgenstern utility function $u(c) = -e^{-\alpha c}$ and date 0 wealth of w holds a portfolio of these two assets. For simplicity, we will allow consumption at date 1 to take negative values.

- (a) Calculate the investor's expected utility if she holds a proportion x of her date 0 wealth in the risky asset.

Hint: If $\tilde{y}$ is normally distributed with mean μ and variance ϕ^2 then

$$E[e^{-\alpha \tilde{y}}] = e^{-\alpha \mu + \frac{1}{2} \alpha^2 \phi^2}.$$

- (b) What level of x maximizes her expected utility?
- (c) How does her portfolio change as her Arrow-Pratt measure of absolute risk aversion increases?
- (d) Now suppose that the investor also has normally-distributed labor income $\tilde{L}$ at date 1, with mean $\bar{L}$, variance θ^2 , and covariance with the risky asset's payoff equal to $\rho\sigma\theta$. That is, her date 1 consumption equals the sum of her portfolio payoff and her labor income. Derive the expected utility maximizing portfolio x and discuss how it is affected by labor income.