

HOMEWORK FOR WEEK OF FEB 8 SELECTED SOLUTIONS

- 3.1 Problem 1c) and the corresponding part of Problem 2.

Write $f_n(z) = \sin \frac{z}{n}$. The pointwise limit $f(z) = \lim_{n \rightarrow \infty} f_n(z)$ is $f(z) = 0$ since $\lim_{n \rightarrow \infty} \sin \frac{z}{n} = \sin(\lim_{n \rightarrow \infty} \frac{z}{n}) = \sin(0) = 0$ for all z . We claim that this convergence is uniform on $\alpha \leq |z| \leq \beta$ for all $0 \leq \alpha < \beta < \infty$ but is not uniform on any annulus of the form $\alpha \leq |z|$ (the $\beta = \infty$ case).

To show uniform convergence in the first case ($\beta < \infty$) we simply have to note that $\sin(z)$ is continuous at 0 and thus for all $\epsilon > 0$ there is a $\delta > 0$ such that $|\sin(z) - 0| < \epsilon$ for all $|z| < \delta$. Letting N_0 be large enough so that $\frac{\beta}{N_0} < \delta$ we have for any $n \geq N_0$ that $|\frac{z}{n}| \leq \frac{\beta}{N_0} < \delta$ for all $|z| < \beta$ and hence that $|f_n(z) - f(z)| = |\sin \frac{z}{n} - 0| < \epsilon$, as required.

For $\beta = \infty$ we have that this sequence does not converge uniformly. Indeed, for each n and any $k \in \mathbb{N}$ with $nk > \frac{\alpha}{\pi}$ we have $nk\pi > \alpha$ and $f_n(nk\pi) = \sin(\pi k) = 0$. It is thus not possible for the sequence $f_n(z)$ to converge uniformly to $f(z) = 1$ on this region.

- 3.1 Problem 1d) and the corresponding part of Problem 2.

Write $f_n(z) = \frac{1}{1+(nz)^2}$; we note that the pointwise limit for $z \neq 0$ is $f(z) = 0$. We claim that convergence is uniform on any subset of the form $\alpha < |z|$ for $\alpha > 0$ and is not uniformly continuous on any subset of the form $0 \leq |z| \leq \beta$.

We show uniform convergence first; for $\alpha < |z|$ we have for sufficiently large n (we require $n\alpha > 1$) that

$$|1 + (nz)^2| \geq |1 - |(nz)^2|| \geq 1 - n^2\alpha$$

so that

$$|f_n(z)| \leq \frac{1}{1 - n^2\alpha}$$

which implies uniform convergence since we thus have for all $\alpha < |z|$ that

$$\lim_{n \rightarrow \infty} |f_n(z) - 0| = \lim_{n \rightarrow \infty} |f_n(z)| \leq \frac{1}{1 - n^2\alpha} = 0.$$

Indeed, this implies *uniform* convergence since this bound is independent of z and depends only on α .

The convergence is not uniform on any region of the form $0 \leq |z| \leq \beta$ (for $\beta < \infty$ or $\beta = \infty$). Indeed, we have that each $f_n(z)$ is continuous at $z = 0$ with $f_n(0) = 1$; one can thus find for each n some point z_n near zero with $|f_n(z_n)| > \frac{1}{2}$ which prevents the $f_n(z)$ from converging uniformly to $f(z) = 0$. Also, I mean, each $f_n(z)$ has poles on this region as well; that's gonna mess things up too.

- 3.1 Problem 4 a)

For $n > 1$ we have that $\oint_{\mathcal{C}} f_n(z) dz = 0$ by Cauchy's Theorem so that $\lim_{n \rightarrow \infty} \oint_{\mathcal{C}} f_n(z) dz = 0$. On the other hand, $f(z) = \lim_{n \rightarrow \infty} \frac{1}{z-n} = 0$ since the denominator becomes arbitrarily large in absolute value as n goes to ∞ so that $\oint_{\mathcal{C}} f(z) dz = 0$.

The above being said, this equality is assured if we can show that $f_n(z)$ converges uniformly to $f(z) = 0$ on some region containing $\mathcal{C}$, say the disc $|z| \leq 1$. Indeed, for all $n > 1$ we have (via an application of the inverse triangle inequality similar to that used in 1d)) that

$$\left| \frac{1}{z-n} \right| \leq \frac{1}{n-1}$$

so that, indeed, $f_n(z)$ converges to $f(z) = 0$ uniformly on this interval.

- 3.1 Problem 5 c)

Yuck, $\text{sech}(z)$, amirite. Still, this isn't too bad to work out via the Weierstrass M test. Since $\text{Re}(z) \geq 1$ we have that

$$|\cosh(nz)| = \left| \frac{e^{nz} + e^{-nz}}{2} \right| \geq \frac{||e^{nz}| - |e^{-nz}||}{2} = \frac{e^{n\text{Re}(z)} - e^{-n\text{Re}(z)}}{2}.$$

We have $e^{n\text{Re}(z)} > 2e^{-n\text{Re}(z)}$ for each n and therefore

$$\frac{e^{n\text{Re}(z)} - e^{-n\text{Re}(z)}}{2} \geq \frac{e^{n\text{Re}(z)}}{4} \geq \frac{e^n}{4}.$$

Finally we have that

$$\text{sech}(nz) \leq 4e^{-n}$$

and

$$\sum_{n=1}^{\infty} 4e^{-n} = \frac{4e^{-1}}{1 - e^{-1}}$$

so that the series converges uniformly by the Weierstrass M test.

- 3.2 Problem 1 c)

Here we have that the radius of convergence is zero. Indeed, this follows from the ratio test since, for $a_n = n^n$ we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{n+1}}{n^n} = (n+1) \left(\frac{n+1}{n} \right)^n \geq n+1$$

so that $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \infty$.

- 3.2 Problem 6 a)

The power series for $\sin(z)$ is of the form $z + z^3g(z)$ for a power series $g(z)$ with radius of convergence ∞ . It follows that $\frac{\sin(z)}{z}$ agrees with the entire function $1 + z^2g(z)$ on $\mathcal{C}$ so that

$$\oint_{\mathcal{C}} \frac{\sin(z)}{z} dz = \oint_{\mathcal{C}} (1 + z^2g(z)) dz = 0$$

by Cauchy's Theorem.

- 3.4 Problem 3

I'll break this into the parts I suggested for you.

a) For $|z| \leq R$ we have

$$\left| \frac{1}{e^n n^z} \right| = (e^{-1})^n |e^{-z \log(n)}| = \frac{1}{e^n} |e^{-\operatorname{Re}(z) \log(n)}| = (e^{-1})^n n^{-\operatorname{Re}(z)} \leq (e^{-1})^n n^R$$

Letting $N_0 \in \mathbb{N}$ be large enough that $n^R \leq e^{\frac{n}{2}}$ for $n \geq N$ we have that

$$(e^{-1})^n n^R \leq (e^{-1})^n e^{\frac{n}{2}} = e^{-\frac{n}{2}} = \left(e^{-\frac{1}{2}} \right)^n$$

Since $\sum_{n=1}^{\infty} \left(e^{-\frac{1}{2}} \right)^n$ converges given that $|e^{-\frac{1}{2}}| < 1$ we have that the series converges by the Weierstrass M test.

b) We have $b_n(z) = e^{-n} e^{-z \log(n)}$ which is clearly an entire function given that e^z is.

c) Let $f_n(z) = \sum_{k=1}^n b_k(z)$. Then $f_n(z)$ is analytic for each n by part b) and for $f(z) = \sum_{n=1}^{\infty} b_n(z) = \lim_{n \rightarrow \infty} f_n(z)$ we have that $f_n(z)$ converges uniformly to $f(z)$ on the region $|z| \leq R$ by part a). Since $f_n(z)$ converges uniformly to $f(z)$ for $|z| < R$ and each $f_n(z)$ is analytic we have that $f(z)$ is analytic for $|z| < R$ by Theorem 3.4.4.