

HOMEWORK – Assigned Dec 1

1. Consider the dispersive wave equation

$$u_{tt} - c^2 u_{xx} + m^2 u = 0, t > 0, -\infty < x < +\infty, c, m \text{ constants},$$

1a) Find solutions of the form

$$u(x, t; k) = \exp[i(kx - \omega(k)t)],$$

for any real k .

1b) Calculate the dispersion relation $\omega(k)$, the phase velocity $v_{ph}(k)$, and the group velocity $v_{gp}(k)$. (Note for each k there will be two dispersion relations $\omega(k) = \pm \dots$.)

1c) Write down the general solution of this wave equation, as an integral over k . Note, because of the two dispersion relations, the general solution will be of the form

$$u(x, t) = \int_{-\infty}^{+\infty} \left[a_+(k) \exp[i(kx + \omega(k)t)] + a_-(k) \exp[i(kx - \omega(k)t)] \right] dk.$$

1d) Write down the unique solution of this wave equation, as an integral over k , that satisfies the initial data $u(x, t=0) = U_{in}(x); u_t(x, t=0) = V_{in}(x)$.

1e) From this Fourier representation of the solution of the initial value problem, describe the approximate behaviour of $u(x, t)$ as $x \rightarrow \infty, t \rightarrow +\infty, x/t = v_{gr}$ held constant.

2. Consider the nonlinear transport equation

$$u_t + uu_x = 0, t > 0, -\infty < x < +\infty,$$

with the initial condition

$$u(x, t=0) = -1, x \leq 0; = 0, x > 0;$$

- 2a)** Write down the ordinary differential equations for the characteristics, including their initial values; and solve these ode's explicitly for the characteristic curves.
- 2b)** Use these characteristics to construct the solution of this initial value problem, and show the triangle in the X-T plane for which the solution is undefined.
- 2c)** Change the initial data to

$$u(x, t = 0) = -1, x \leq -\epsilon; = x/\epsilon, -\epsilon < x < 0; = 0, 0 < x,$$

where the constant $\epsilon > 0$.

Construct the solution of this initial value problem, and show there is no triangle in the X-T plane for which the solution is undefined.

2d) Consider the limit as $\epsilon \rightarrow 0$ through positive values of ϵ . Plot the limit, and conjecture how the triangle of part **2b** should be "filled in".

3. Consider the nonlinear transport equation

$$u_t + uu_x = 0, t > 0, -\infty < x < +\infty,$$

with the initial condition

$$u(x, t = 0) = +1, x \leq -\epsilon; = -x/\epsilon, -\epsilon < x < 0; = 0, 0 < x.$$

- 3a)** Carefully sketch the characteristic curves.
- 3b)** Use these characteristics to construct the solution of this initial value problem, and show where in the X-T plane that solution is defined (i.e., where for fixed (x, t) , there exists exactly one characteristic through (x, t)).
- 3c)** Show for which (x, t) in the (X, T) plane there are multiple (more than one) characteristic through (x, t) [how many]. (For this region, the solution is undefined, and something extra must be added to define it uniquely.)