About polynomial interpolation and comparison

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1 Polynomial approximation of $f(x)$

- using a nth order polynomial to approximate a function $f(x)$

1.1 Taylor series at x_0

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots \tag{1}$$

1.2 Lagrange polynomial interpolation

let $t_j, j = 0 : n$ are n interpolation points, we want

$$p(t_j) = s(t_j), \;\; j = 0, ..., n \tag{2}$$

then

$$p(x) = \sum_{j}^{n} s(t_j) l_j(x), \quad l_j(x) = \frac{\Pi_{k \neq j}(x - t_k)}{\Pi_{k \neq j}(t_j - t_k)} \tag{3}$$

2 Numerical experiments

2.1 approximation of $f(x) = \frac{1}{|x|}, \;\; x \in [2, 3]$

- Taylor approximation. Let x_0 be the center of the interval $[2, 3]$.
  - plot 4 figures as in page 1,2 of [1] for $f(x) = 1/x$ at $x = 2.5$

$$p_n(x) = \frac{1}{2.5} + f'(2.5)(x - 2.5) + \frac{f''(2.5)}{2}(x - 2.5)^2 + \frac{f''(2.5)}{3!}(x - 2.5)^3 + \cdots \tag{4}$$

where

$$f'(x) = -\frac{1}{x^2} \Rightarrow f'(2.5) = -\frac{1}{2.5^2} \tag{5}$$

$$f''(x) = \frac{1}{x^3} \Rightarrow f''(2.5) = \frac{1}{2.5^3} \tag{6}$$

i.e. $p_1(x) = \frac{1}{2.5}$

  - plot $|f(x) - p_n|$ for different n

- Measure the error at $z = 2 : h : 3$ with $h = 0.01$, plot the relative error in 2 norm versus n, where relative error in 2 norm is

$$E = \frac{\sqrt{\sum_{i=1}^{N}(f(z_i) - p_n(z_i))^2}}{\sqrt{\sum_{i=1}^{N} f(z_i)^2}} \tag{7}$$

- Interpolation.
  - Read notes [1] page 3 - 5, especially the examples
  - Write down the Lagrange interpolation formula for $n = 2$ and $n = 3$.
    * $n = 2 : p_2(x) = \sum_{j=0}^{2} f(x_j)l_j(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x)$ where

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \tag{8}$$
$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \tag{9}$$
$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \tag{10}$$

    * $n = 3$?
  - write a code to plot the $p_n(x)$ to approximate $f(x) = \frac{1}{x}$ in $[2.0, 3.0]$ and the error for $n = 2, 3$ with uniform interpolation points, just the same as we did for Taylor approximation. For example, when
    * $n = 2, x_0 = 2.0, x_1 = 2.5, x_2 = 3.0$;
    * $n = 3, x_0 = 2.0, x_1 = 2.0 + 1/3, x_2 = 2.0 + 2/3, x_3 = 3.0$.
    * in general, $x_i = x_0 + hi, i = 0 : n$ where $h = \frac{x_n - x_0}{n}$. In our problem, $x_0 = 2, x_n = 3, h = \frac{1}{n}$
    * more general, for interpolation in a $[a, b]$, $x_i = a + \frac{b-a}{n}i, i = 0 : n$
  - Modify your code, to make it work for any n
  - plot relative error versus n, similar to the Taylor series.
  - Chebyshev (2nd kind) interpolation points
    * $x_i = -\cos\theta_i,\ \ \theta_i = ih, h = \pi/n,\ \ i = 0 : n$. Note that $x_i \in [-1, 1]$
    * we need to map the Chebyshev points in [-1,1] to an arbitrary domain $[a, b]$. For example, $x_i \in [-1, 1,], t_i \in [a, b]$

$$\frac{x_i - (-1)}{2} = \frac{t_i - a}{b - a} \quad \Rightarrow \quad t_i = a + (b - a)\frac{x_i + 1}{2} \tag{11}$$

  - Let's consider $x \in [2, 3]$ to avoid the singularity. Map the $n+1$ Chebyshev interpolation points from [-1,1] to $[2, 3]$. Find the Lagrange polynomial interpolation $L_n(x)$ with 1) Uniform points 2) Chebyshev points. Measure the error at $z = 2 : h : 3$ with $h = 0.01$, plot the relative error in 2 norm versus n. You may not see big difference for function $1/x$.
  - try Runge function $\frac{1}{1+25x^2}, x \in [-1, 1]$ for practice purpose.
- Cubic spline

2.2 functions in Fig. 2 of [2]

References

[1] L.Wang, chapter_3_notes.pdf

[2] L.N. Trefethen (2008), Is Gauss Quadrature Better than Clenshaw-Curtis *SIAM Rev.* **50**(1), 67–87