

## INSTRUCTIONS

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- Answer **ALL** questions
- The weight of this paper is 70%.
- Be sure to write your ID number on your submitted paper or papers
- This is an “open book-take home exam”
- Please **pdf** your solutions (scanned or typed) before uploading.
- Name your pdf file(s) in the format **IDnumber.pdf**.

For example,

6200987654.pdf

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# Question 1 (15 marks)

- (a) Gambler Shan has \$3. Her objective is to convert this amount, through gambling to a fortune of \$8. She bets against her friend Scam. If Shan bets \$ $x$ , she wins \$ $x$  with probability  $p = 0.4$  and loses \$ $x$  with probability  $q = 0.6$ . Your job is to advise Shan on which of two strategies to employ.

- **Strategy One:** Bet \$1 each time.
- **Strategy Two:** With each bet, bet as much as she can that brings her fortune to or towards \$8 (without exceeding \$8)

(Note a fortune of \$5 means her net winnings is \$2)

Which of the two strategies should Shan employ?

Give full justification.

[10]

- (b) The matrix

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/7 & 6/7 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the transition probability matrix of a Markov chain  $\{X_n : n \in \mathbb{Z}^+ \cup \{0\}\}$  with state space  $S = \{0, 1, 2, 3, 4\}$ .

- (i) Draw the transition diagram (chain) that  $P$  represents [1]
- (ii) What are the communicating classes and classify them as recurrent (closed) or transient (open)? [2]
- (iii) What is  $\lim_{n \rightarrow \infty} P(X_n = 0 | X_0 = 4)$ ? [2]

## Question 2 (25 marks)

- (a) The arrival of customers to a service arrive station can be modeled as a Poisson process with arrival rate 6 customers per hour. Servicing of customers starts at time  $T$  (in hours) where  $T$  is the time for the arrival of the **third** customer.

(i) What is  $E(T)$ ? [2]

(ii) What is  $P(T > 1)$  [3]

- (b) A birth and death process  $\{N(t)\}$  has three states  $\{0, 1, 2\}$ . With the usual notation,

$$\lambda_k = \alpha(2 - k) \text{ and } \mu_k = \beta k \text{ for } k \in \{0, 1, 2\}.$$

(i) Draw the rate-state transition diagram. [3]

(ii) Determine the (long term) steady-state distribution of  $\{N(t)\}$ . [7]

- (c) The time to failure of two machines operating simultaneously and independently of each other are such that they both are exponentially distributed with mean failure time  $1/\mu$ . Repair times are exponentially distributed with rate  $\lambda$ .

Let  $N(t)$  be the number of machines in operation at time  $t$ .  $\{N(t)\}$  may be modeled as a 3-state birth and death process (queueing model)

(i) Identify the state space  $S$  and construct the rate generator matrix  $G$ . [3]

(ii) Let  $P_n = P(N(t) = n)$ , the long term probability that  $n$  machines are in operation at a given time in the distant future.

Determine  $(P_0, P_1, P_2)$  in terms of  $\lambda$  and  $\mu$  and confirm that in the case where  $\lambda = \mu = 1$ ,  $P_0 = 2/5$ . [7]

## Question 3 (14 marks)

A rastaman barber who lives in Papine has shop space for 4 waiting customers (can accommodate 5 patrons). The arrival of potential customers follow a Poisson distribution with a rate of 5 per hour. When the facility is full, arriving customers leave without entering. The time to cut a hair has an exponential distribution with a mean of 15 minutes.

- (i) What is the probability that a newly arriving customer can get their hair cut without waiting on the rasta barber? [2]
- (ii) What is the expected number of customers waiting in the shop, waiting on the rasta barber? [5]
- (iii) How much time on average do customers spend in the barber shop? [5]
- (iv) What fraction of potential customers are “turned away”? [2]

## Question 4 (16 marks)

The steady-state probabilities of a two-server queueing model are

$$P_0 = \frac{1}{16}, P_1 = \frac{1}{4}, P_2 = \frac{3}{8}, P_3 = \frac{1}{4}, P_4 = \frac{1}{16}, P_n = 0 \ (n \geq 5)$$

For this model, the arrival rates are

$$\lambda_0 = 8, \lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1, \lambda_n = 0 \ (n \geq 3)$$

- (i) What is the expected queue length? [3]
- (ii) At a given time, what is the expected number of customers being served? [3]
- (iii) Per customer, what is the expected time spent in the queue? [5]
- (iv) On average, what is the expected time spent to serve a customer. [5]

**END OF EXAMINATION PAPER**