Math 1019N Final Exam Review

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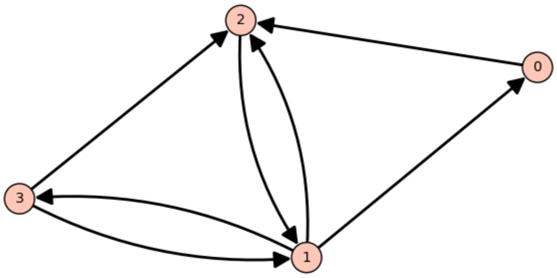
Question 1.

(a) Compute the Adjacency matrix of the following Digraph.



Solution:

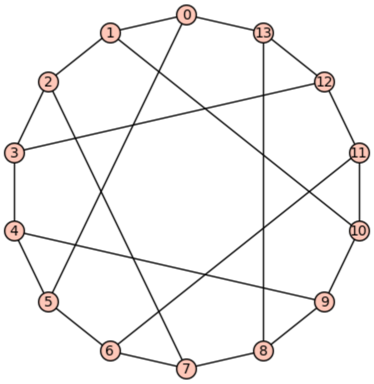
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

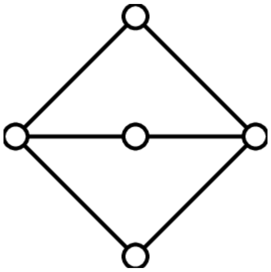
(b) Does a digraph always have to have zeros along the diagonal of it's adjacency matrix?

Solution:
No, notice that I did not say simple and so if any vertex had a loop then it would be adjacent to itself and have 1 in the corresponding entry of the adjacency matrix.

Question 2.

Determine whether the following graph has a Hamilton circuit or Hamilton path.





Question 3.

Prove that the integers mod 5 is an equivalence relation

$$x \equiv y\ (mod 5) \text{ if and only if } \frac{x-y}{5} \in \mathbb{Z}$$

Reflexivity

$$x - x = 0$$
$$\frac{0}{5} = 0 \in \mathbb{Z}$$
$$\therefore\ x \equiv x\ (mod\ 5)$$

Symmetry

Let x, y be congruent to one another modulo 5.

Then

$$x \equiv y\ (mod\ 5)$$

is equivalent to saying

$$\frac{x-y}{5} \in \mathbb{Z}$$

Notice that if any integer is divisible by 5 it is also divisible by -5 and so we can conclude that

$$\frac{y-x}{5} \in \mathbb{Z}$$

and thus

$$x \equiv y\ (mod\ 5) \iff y \equiv x\ (mod\ 5)$$

Transitivity

Suppose we have x, y, z such that

$$x \equiv y\ (mod\ 5) \text{ and } y \equiv z\ (mod\ 5)$$

$$x \equiv y\ (mod\ 5) \rightarrow \frac{x-y}{5} = k \in \mathbb{Z}$$
$$y \equiv z\ (mod\ 5) \rightarrow \frac{y-z}{5} = l \in \mathbb{Z}$$
$$\frac{x-5k-z}{5} = l \iff \frac{x-z}{5} = l + k \in \mathbb{Z}$$
$$\therefore x \equiv z\ (mod\ 5)$$

Question 4.

Prove by mathematical induction that the sum of the first n integers is equal to

$$\frac{n(n+1)}{2}$$

Define P(n):
Let P(n) be the following statement

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base Case P(1):
Notice that the sum of the first integer is just 1 and thus computinging

$$\frac{n(n+1)}{2}$$

where n = 1, gives us $\frac{1(1+1)}{2} = 1$. Therefore the LHS = RHS and our base case holds.

Inductive Hypothesis:
Some for some arbitrary fixed integer $k > 1$ that the statement $P(k)$ is true. Use this statement to show that $P(k) \to P(k+1)$

Inductive Step:
The LHS of the statement P$(k+1)$ is as follows

$$\sum_{i=1}^{k+1} i$$

We can break this sum up into the sum of the first k integers plus the $k+1$ term

$$\sum_{i=1}^{k+1} i = (k+1) + \sum_{i=1}^{k} i$$

By our inductive hypothesis we know that the sum of the first k integers is equal to $\frac{k(k+1)}{2}$ and so we get

$$\sum_{i=1}^{k+1} i = (k+1) + \frac{k(k+1)}{2}$$

Placing the $(k+1)$ over a common denominator and expanding $k(k+1)$ we get.

$$\sum_{i=1}^{k+1} i = \frac{k^2+3k+2}{2}$$

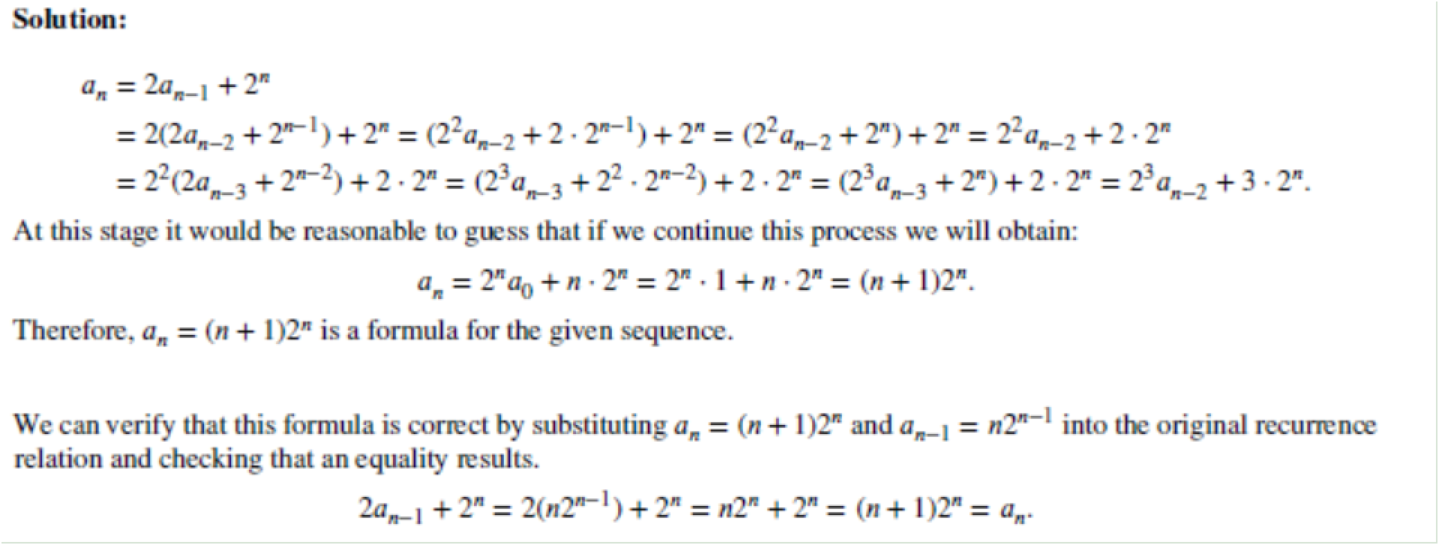
And then factoring by the quadratic equation we get the form

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Which is exactly the statement P$(k+1)$ and therefore we have proven our statement P(n) to be true for all n by mathematical induction.

Question 5.

Find a formula for the recurrence relation $a_n = 2 * a_{n-1} + 2^n, a_0 = 1$



Solution:

$$\begin{aligned} a_n &= 2a_{n-1} + 2^n \\ &= 2(2a_{n-2} + 2^{n-1}) + 2^n = (2^2 a_{n-2} + 2 \cdot 2^{n-1}) + 2^n = (2^2 a_{n-2} + 2^n) + 2^n = 2^2 a_{n-2} + 2 \cdot 2^n \\ &= 2^2(2a_{n-3} + 2^{n-2}) + 2 \cdot 2^n = (2^3 a_{n-3} + 2^2 \cdot 2^{n-2}) + 2 \cdot 2^n = (2^3 a_{n-3} + 2^n) + 2 \cdot 2^n = 2^3 a_{n-2} + 3 \cdot 2^n. \end{aligned}$$

At this stage it would be reasonable to guess that if we continue this process we will obtain:

$$a_n = 2^n a_0 + n \cdot 2^n = 2^n \cdot 1 + n \cdot 2^n = (n + 1)2^n.$$

Therefore, $a_n = (n + 1)2^n$ is a formula for the given sequence.

We can verify that this formula is correct by substituting $a_n = (n + 1)2^n$ and $a_{n-1} = n2^{n-1}$ into the original recurrence relation and checking that an equality results.

$$2a_{n-1} + 2^n = 2(n2^{n-1}) + 2^n = n2^n + 2^n = (n + 1)2^n = a_n.$$

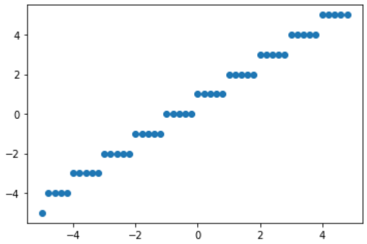
Question 6.

Let f be the following map $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \lceil x \rceil$$

Is f a function? Is it one to one and onto? What is the domain, codomain and range of the f?

Solution
Yes it is a function. If you were to plot it, it would something like



Clearly it passes the vertical line test and is well defined as so it is a function. The Domain and Codomain of the function is the real numbers as that is what I defined it to be at the beginning of the question. However it's range is the integers. No matter what real number from the domain I put in the ceiling function rounds up to the next integer.

The function is not one to one. Note that both $f(1.5)$ and $f(2)$ are equal to 2 in this function (What would the inverse image of $\{2\}$ be?). We have already seen that the function is not onto as well as the codomain and the range are not equal.

Question 7.

Prove that if A, B are sets then $P(A) \cup P(B) \subseteq P(A \cup B)$. Where $P(A)$ is the power set of A.

Proof:

Let X be an arbitrary element of $P(A) \cup P(B)$. Then by the definition of union X is either an element of $P(A)$ or an element of $P(B)$. Break into two cases.

Case 1: $X \in P(A)$

If $X \in P(A)$ then by definition of the power set. $X \subseteq A$. This implies that X is a subset of $A \cup B$, which by definition of the power set says $X \in P(A \cup B)$.

Case 2: $X \in P(B)$

If $X \in P(B)$ then by definition of the power set. $X \subseteq B$. This implies that X is a subset of $A \cup B$, which by definition of the power set says $X \in P(A \cup B)$.

Therefore we have shown that any arbitrary element of $P(A) \cup P(B)$ must also be inside of $P(A \cup B)$ and so $P(A) \cup P(B) \subseteq P(A \cup B)$.

Question 8.

Use the truth table to determine if the following argument is valid.

$$\begin{array}{c} p \to q \\ q \to r \\ \therefore\ p \to r \end{array}$$

Solution:

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Row 1 and row 5 are the only scenarios in which all premises are true. Row 1 and 5 also returns that our conclusion is true and thus the argument is valid.

Question 9.

Prove that $0.333\ldots = \frac{1}{3}$, by using the geometric sequence.

Solution: Note that the first thing we can do with $0.333\ldots$ is break it down into a sum of infinite fractions

$$0.333\ldots = 0.3 + 0.03 + 0.003\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}\ldots$$

In each term of the sum we have the numerator staying the same and the denominator growing by a factor of 10. This we can use for our a and r terms.

$$0.333\ldots = \frac{3}{10} + (\frac{3}{10})(\frac{1}{10}) + (\frac{3}{10})(\frac{1}{10})^2\ldots$$

$$0.333\ldots = \sum_{n=0}^{\infty}(\frac{3}{10})(\frac{1}{10})^n$$

We have shown that $0.333\ldots$ is equal to a geometric series with $a = \frac{3}{10}$ and $r = \frac{1}{10}$. Since we have that $r < 1$ we know it converges and is equal to.

$$0.333\ldots = \sum_{n=0}^{\infty}(\frac{3}{10})(\frac{1}{10})^n = \frac{\frac{3}{10}}{(1 - \frac{1}{10})}$$

And we simplify,

$$\frac{\frac{3}{10}}{(1 - \frac{1}{10})} = \frac{3}{9} = \frac{1}{3}$$

Question 10.

Prove the following using propositional logic

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q.$$

Proof: Starting from the LHS and applying DeMorgan's Law we get

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$$

Applying DeMorgan's law once again to the smaller conjuction

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge (p \vee \neg q)$$

Now distribute the $\neg p$

$$\neg(p \vee (\neg p \wedge q)) \equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$$

Notice that $(\neg p \wedge p)$ is a contradiction and we then arrive at our desired result

$$\neg(p \vee (\neg p \wedge q)) \equiv F \vee (\neg p \wedge \neg q) = (\neg p \wedge \neg q)$$