

NOOR KHALIF

1. Prove that $3^N < N!$ if N is an integer that exceeds 6

Base Step $n=7$

Since $3^7 < 7! = 2187 < 5040$ hence $P(7)$ is true

Assume $3^N < N!$ then
 $3^{N+1} < (n+1)!$

$$3 \cdot 3^N < (n+1) N!$$

We know that $3^N < N!$ so we need to show that $3 < (n+1)$

$3 < (n+1)$ which is always true since $n > 6$

2. 5 is a divisor of $N^5 - N + 5$

Let $P(N) = N^5 - N$

$P(1) = 0$, this is divisible by 5, the statement is true for $n=1$

$$P(N) = N^5 - N = 5t, \text{ where } t \in \mathbb{Z}$$

Assume that the statement is true for $n=N$

Then

$$\begin{aligned} P(N+1) &= (N+1)^5 - (N+1) \\ &= N^5 + 5N^4 + 10N^3 + 10N^2 + 5N + 1 - N - 1 \\ &= N^5 + 5N^4 + 10N^3 + 10N^2 + 5N \end{aligned}$$

$$\Rightarrow 1 - 1 = 0$$

$$= (N^2 - N) + 5(N^4 + 2N^3 + 2N^2 + N)$$

$$= 5 + 5(N^4 + 2N^3 + 2N^2 + N)$$

$$= 5(1 + (N^4 + 2N^3 + 2N^2 + N))$$

Therefore $P(N+1)$ is divisible by 5, the Statement is true for all $n \in \mathbb{Z}$

3. Let $P(n)$ be the Proposition to prove
Basis Step:

3¢ postage can be formed by three 1¢ stamps
while 8¢ postage can be formed by
two 4¢ stamps

Inductive step:

Assume that for all j such that $8 \leq j \leq 14$,
 j -cents postage can be formed using
only ~~1¢ and~~ 2¢ and 6¢ stamps (proof)

Assume that the inductive hypothesis is true
then $n+1-2$ cents ($= n-1$ cents) postage,

We can form $j = n-2$ cents postage using
only 2¢ and 6¢ stamps whenever
 $8 \leq j \leq 14$

Therefore, as long as $j = n-2 \geq 14$ the
inductive hypothesis holds

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1, 1, 2, 6, 24, 120, 720, ...

$$1! = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

* Relation is $N!$ for $N = 1, 2, 3, \dots, \infty$

OR

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$$1 \times 2 = 2$$

$$2 \times 3 = 6$$

$$6 \times 4 = 24$$

$$24 \times 5 = 120$$

$$120 \times 6 = 720$$

$$720 \times 7 = 5040$$

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✓ Every nonempty set S of non-negative integers contains at least ~~at~~ element, i.e. there is some integer a in S such that $a \leq b$, $a \leq b$ ~~for~~

for all b 's belonging

✓ Every non-empty set of positive integers contains a least element.

Rational numbers do not form a well-ordering under the usual less-than relation

Unlike the standard ordering $\leq$ of the natural numbers, the standard ordering $\leq$ of the integers is not well ordering. E.g. the set of negative integers does not contain a least element.

6

For any positive integer N , $4^{N+1} + 5^{2N-1}$ is a multiple of 21

Basis Step

$$S(1) : 4^{(1)+1} + 5^{2(1)-1} = 4^2 + 5^1 = 16 + 5 = 21, \text{ which is divisible by } 21$$

Inductive Step :

Assume $S(k)$ is true, i.e. assume that $4^{k+1} + 5^{2k-1}$ is divisible by 21

$$\Rightarrow 4^{k+1} + 5^{2k-1} = 21A; A \in \mathbb{N}$$

$$S(k+1) : 4^{(k+1)+1} + 5^{2(k+1)-1}$$

$$= 4^{k+2} + 5^{2k+2-1}$$

$$= 4^{k+2} + 5^{2k+1}$$

$$= 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1}$$

$$= 4 \cdot 4^{k+1} + (21+4) \cdot 5^{2k-1}$$

$$= 4 \cdot 4^{k+1} + 21 \cdot 5^{2k-1} + 4 \cdot 5^{2k-1}$$

$$= 4(4^{k+1} + 5^{2k-1}) + 21 \cdot 5^{2k-1}$$

$$= 4 \cdot 21A + 21 \cdot 5^{2k-1}$$

$$= 21 \cdot 4A + 21 \cdot 5^{2k-1}$$

$$= 21(4A + 5^{2k-1}) ; \text{ which is divisible by 21}$$

So $S(k+1)$ is true whenever $S(k)$ is true

Therefore, $4^{n+1} + 5^{2n+1}$ is divisible by 21

Proof: $\therefore 6 \cdot (1^2 + 2^2 + 3^2 + 4^2 + \dots + N^2) = N(N+1)(2N+1)$

Let $P(n): 1^2 + 2^2 + 3^2 + 4^2 + \dots + N^2 = \frac{N(N+1)(2N+1)}{6}$

For $N=1$

L.H.S = $1^2 = 1$

R.H.S = $\frac{1(1+1)(2 \times 1 + 1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$

Hence L.H.S = R.H.S

$\therefore P(N)$ is true for $N=1$

Assume $P(k)$ is true

$$1 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \text{--- (1)}$$

We prove that $P(k+1)$ is true

$$1 + 2^2 + 3^2 + \dots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2 \times (k+1)+1)}{6}$$

$$1 + 2^2 + 3^2 + \dots + (k+1)^2 = \frac{(k+1)(k+2)(2k+2+1)}{6}$$

$$1 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6} \quad \text{--- (2)}$$

We have to prove $P(k+1)$ from $P(k)$ i.e.
② from ①

From (1)

$$1 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Adding $(k+1)^2$ on both sides

$$1^2 + 2^2 + 3^2 + 4^2 + \dots + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2 + k)}{6}$$

$$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 4k + 3k + 6)}{6}$$

$$= \frac{(k+1)(2k(2k+2) + 3(k+2))}{6}$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$$\text{Thus } 1^2 + 2^2 + 3^2 + 4^2 + \dots + k^2 + (k+1)^2$$

$$= \frac{(k+1)(2k+3)(k+2)}{6} \quad \text{which is the same as } P(k+1)$$

$\therefore P(k+1)$ is true when $P(k)$ is true when n is a natural number

8 Show that $f_1^2 + f_2^2 + f_3^2 + \dots + f_n^2 = f_n f_{n+1}$

Solution

Basis step

$$f_1 f_2 = (1) \cdot (1) = 1 = 1^2 = f_1^2 \Rightarrow \text{The proposition is true when } n=1$$

Inductive step

Assume that for some $n \geq 1$ $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$

$$n \geq 1 \quad f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$$

We use the above hypothesis to prove

$$f_1^2 + f_2^2 + f_3^2 + \dots + f_{n+1}^2 = f_{n+1} f_{n+2}$$

Assume that the inductive hypothesis is true ; then

$$\begin{aligned} f_1^2 + f_2^2 + \dots + f_{n+1}^2 &= f_1^2 + f_2^2 + \dots + f_n^2 + f_{n+1}^2 \\ &= f_n f_{n+1} + f_{n+1}^2 \quad (\text{using the inductive hypothesis}) \\ &= f_n \cdot f_{n+1} + f_{n+1} f_{n+1} \\ &= f_{n+1} (f_n + f_{n+1}) \quad (\text{using common factor of } f_{n+1}) \\ &= f_{n+1} f_{n+2} \quad (\text{using recursive definition of } f_{n+2}) \end{aligned}$$
