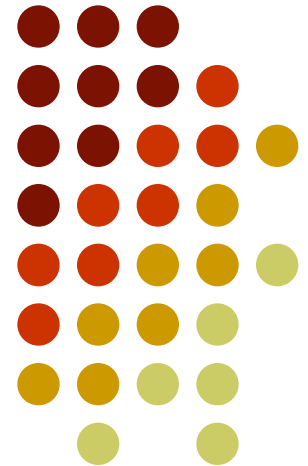


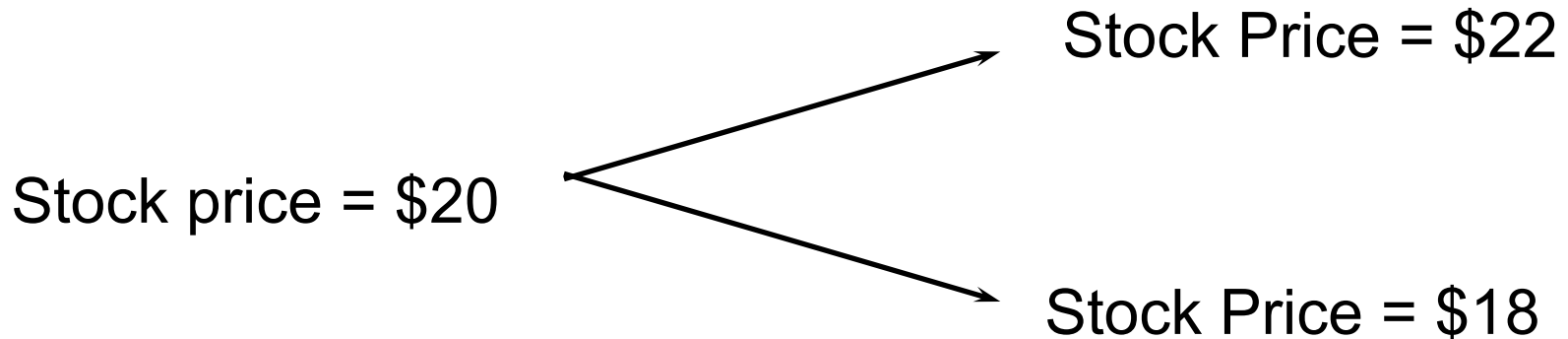
Binomial Trees



A Simple Binomial Model



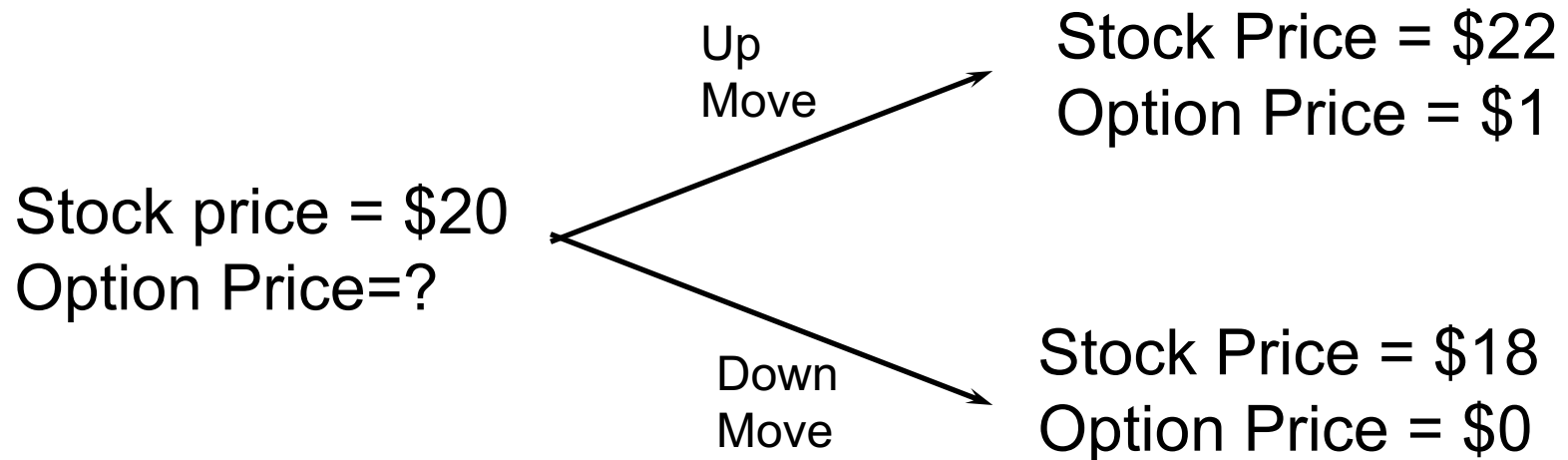
- A stock price is currently \$20
- In three months it will be either \$22 or \$18



A Call Option



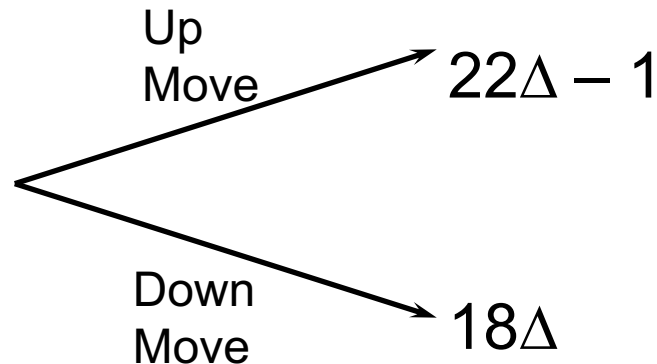
A 3-month call option on the stock has a strike price of 21.





Setting Up a Riskless Portfolio

- For a portfolio that is long Δ shares and a short 1 call option values are



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio

(Risk-Free Rate is 12%)



- The riskless portfolio is:
 long 0.25 shares
 short 1 call option
- The value of the portfolio in 3 months is
$$22 \times 0.25 - 1 = 4.50$$
- The value of the portfolio today is
$$4.5e^{-0.12 \times 0.25} = 4.3670$$



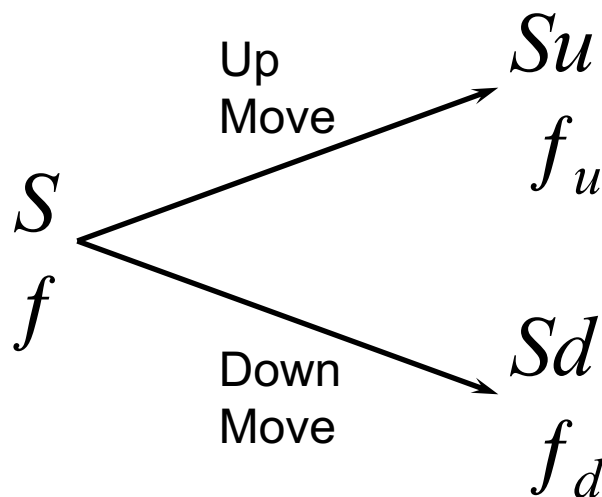
Valuing the Option

- The portfolio that is
 long 0.25 shares
 short 1 option
is worth 4.367
- The value of the shares is
 5.000 ($= 0.25 \times 20$)
- The value of the option is therefore
 0.633 ($= 5.000 - 4.367$)

Generalization



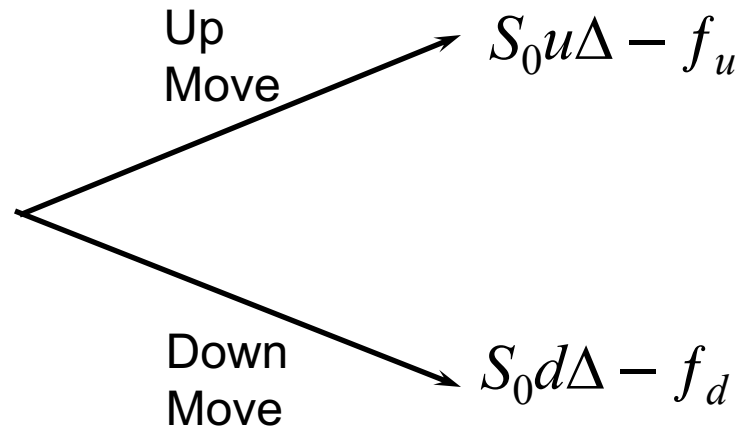
A derivative lasts for time T and is dependent on a stock





Generalization (continued)

- Value of a portfolio that is long Δ shares and short 1 derivative:



- The portfolio is riskless when $S_0 u \Delta - f_u = S_0 d \Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$



Generalization

(continued)

- Value of the portfolio at time T is

$$Su \Delta - f_u$$

- Value of the portfolio today is

$$(Su \Delta - f_u)e^{-rT}$$

- Another expression for the portfolio value today is $S \Delta - f$

- Hence

$$f = S \Delta - (Su \Delta - f_u)e^{-rT}$$

Generalization

(continued)



Substituting for Δ we obtain

$$f = [pf_u + (1 - p)f_d]e^{-rT}$$

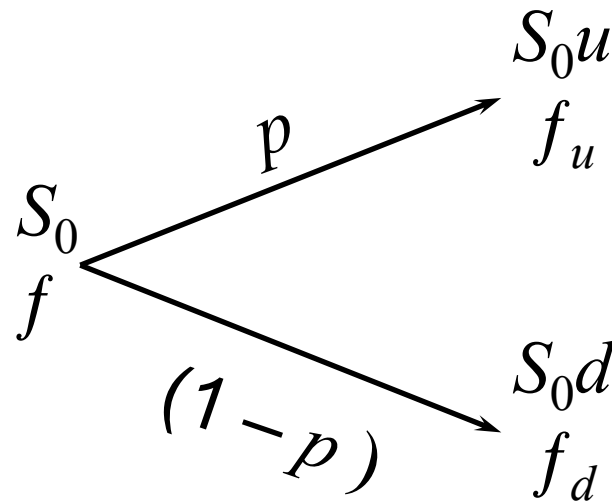
where

$$p = \frac{e^{rT} - d}{u - d}$$



p as a Probability

- It is natural to interpret p and $1-p$ as the probabilities of up and down movements
- The value of a derivative is then its expected payoff in discounted at the risk-free rate





Risk-Neutral Valuation

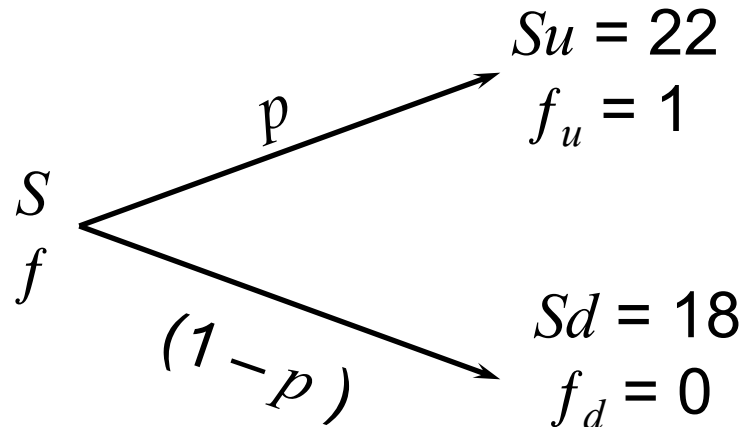
- When the probability of an up and down movements are p and $1-p$ the expected stock price at time T is $S_0 e^{rT}$
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

Irrelevance of Stock's Expected Return



When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant

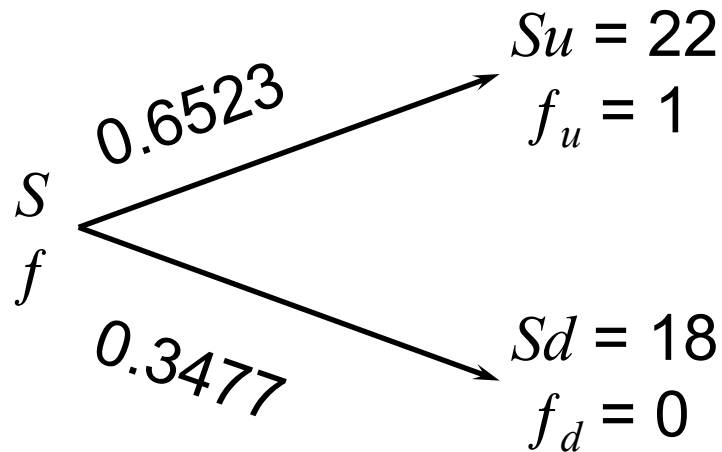
Original Example Revisited



$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

- Since p is a risk-neutral probability
$$20e^{0.12 \times 0.25} = 22p + 18(1-p); p = 0.6523$$
- Alternatively, we can use the formula

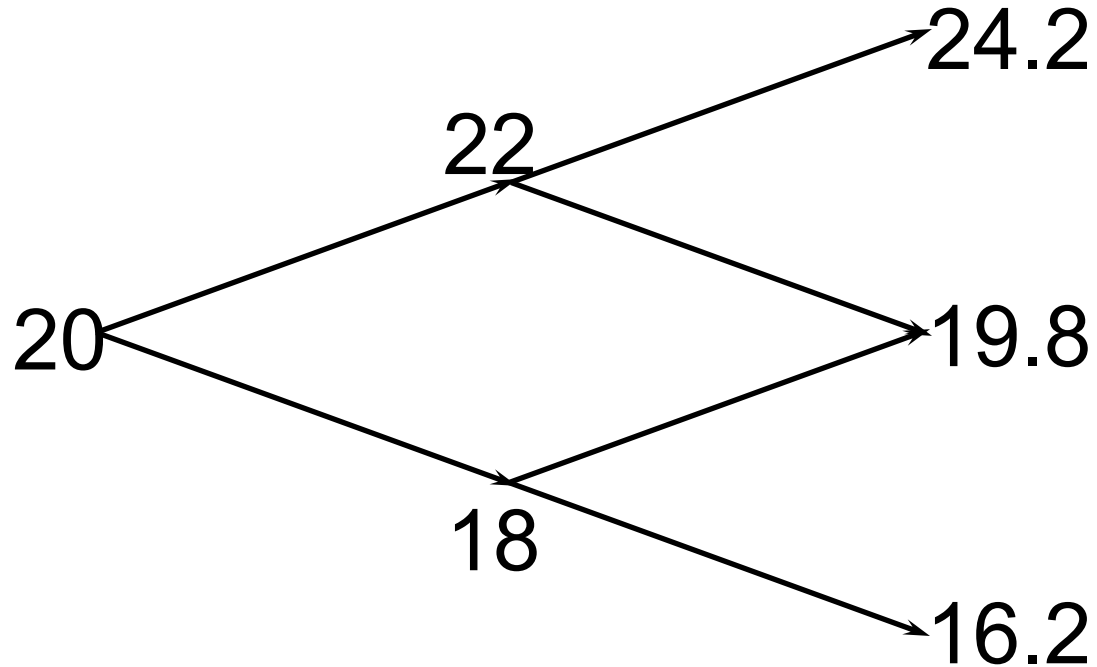
Valuing the Option Using Risk-Neutral Valuation



The value of the option is

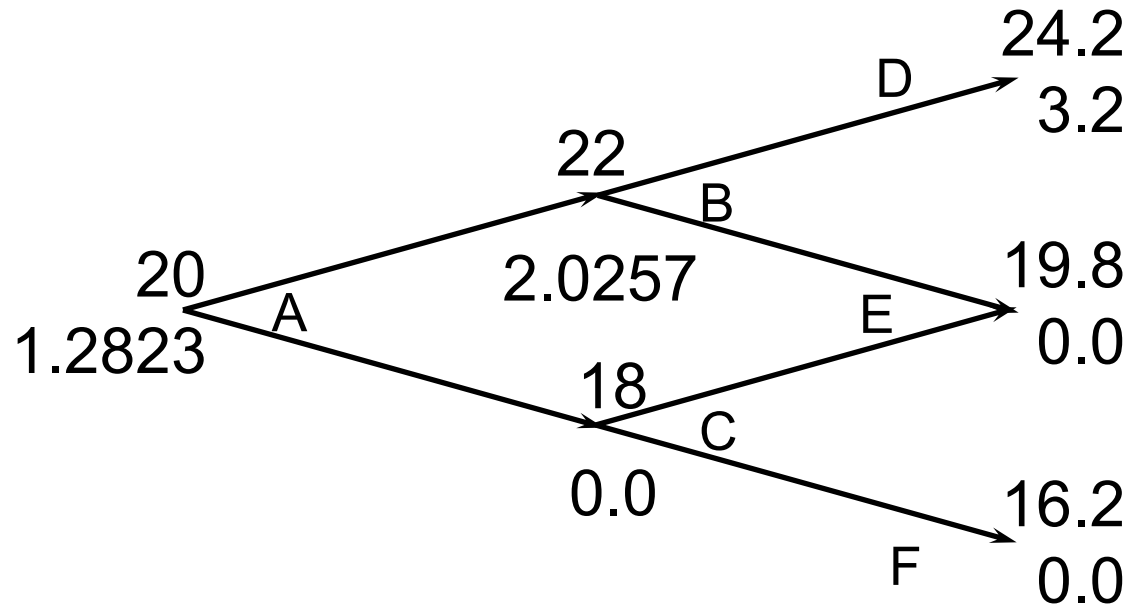
$$e^{-0.12 \times 0.25} [0.6523 \times 1 + 0.3477 \times 0] \\ = 0.633$$

A Two-Step Example



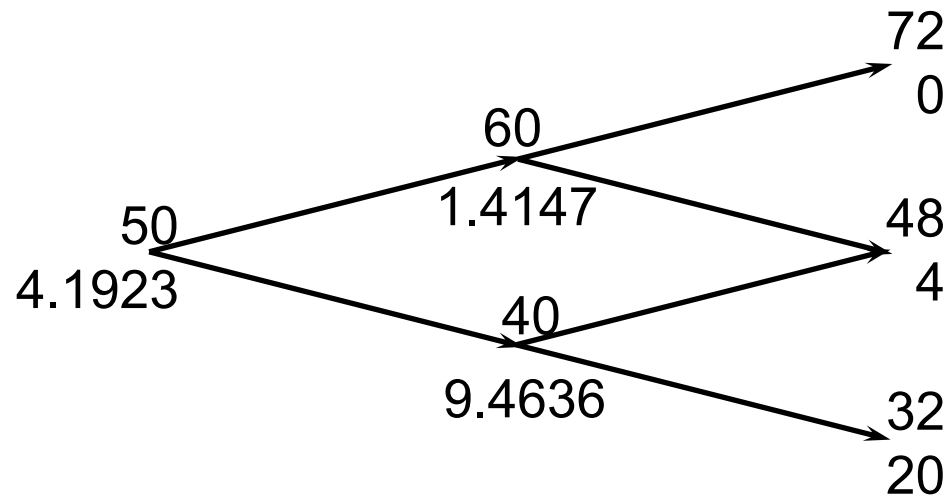
- Each time step is 3 months
- $K=21$, $r=12\%$

Valuing a Call Option



- Value at node B
$$= e^{-0.12 \times 0.25} (0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257$$
- Value at node A
$$= e^{-0.12 \times 0.25} (0.6523 \times 2.0257 + 0.3477 \times 0)$$
$$= 1.2823$$

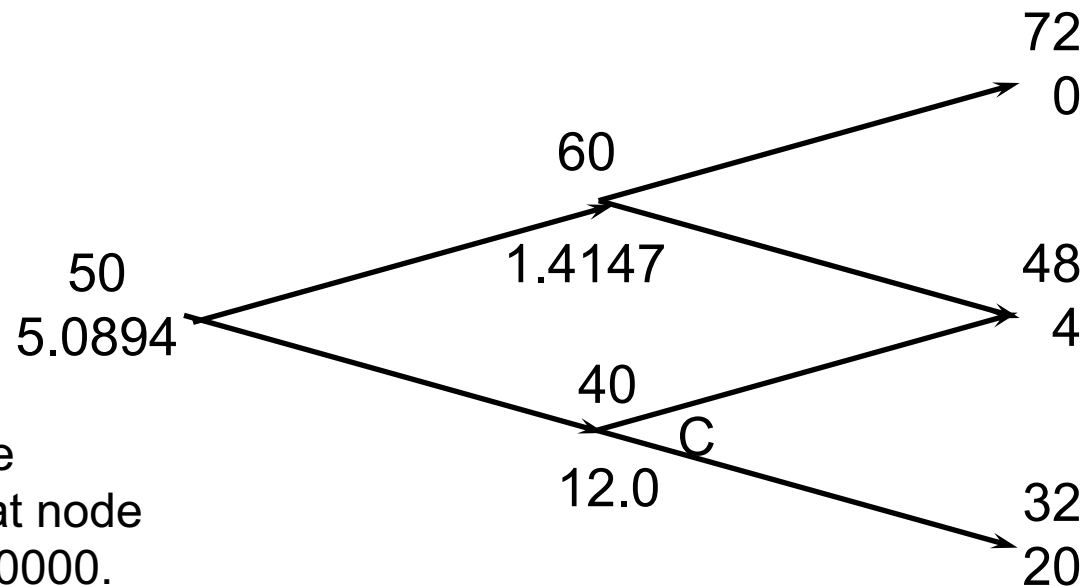
A Put Option Example



$K = 52$, time step = 1yr

$r = 5\%$, $u = 1.32$, $d = 0.8$, $p = 0.6282$

What Happens When the Put Option is American



The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

Delta



- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node



Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein (1979)

Assets Other than Non-Dividend Paying Stocks



- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p



The Probability of an Up Move

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract



Increasing the Time Steps

- In practice at least 30 time steps are necessary to give good option values

The Black-Scholes-Merton Model



- The BSM model can be derived by looking at what happens to the price of a European call option as the time step tends to zero