

LECTURE NOTES ON DYNAMICAL SYSTEMS, CHAOS AND FRACTAL GEOMETRY

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Preface

These notes were created over a number of years of teaching senior seminar type courses to senior year students at Towson University and graduate courses in the Towson University Applied Mathematics Graduate Program. Originally I based the course on some of the existing text books currently available, such as [63], [22], [28] and [20], but gradually realized that these didn't fully suit the needs of our students, so I produced my own lecture notes. However, these notes owe a debt to the above mentioned books. The first 12 chapters cover fairly standard topics in (mostly) one-dimensional dynamics and should be accessible to most upper level undergraduate students. The requirements from real analysis and topology (metric spaces) are developed as the material progresses. Chapter 13 gives a (non-rigorous) introduction to the theory of substitutions (via examples) and we show how these can give rise to certain types of fractals. Subsequent chapters develop the rigorous mathematical theory of substitutions and Sturmian sequences. Although there are numerous expositions of this material, most either give a non-rigorous account with no depth, or assume a level of sophistication above that of the upper level undergraduate or beginning graduate student. Our study of substitutions is purely topological and avoids any measure theory. Consequently we do not touch on some important topics of current interest such as the spectral properties of substitutions (see [49]).

Some of this material owes a debt to quite recent publications in the field of dynamical systems appearing in journals such as the *American Mathematical Monthly*, the *College Mathematics Journal*, *Mathematical Intelligencer* and *Mathematics Magazine*. Various Internet resources have been used such Wolfram's MathWorld, some of these without citation because of difficulties in knowing who the author is. All of the figures were created using the computer algebra system Mathematica. Computer algebra systems are indispensable tools for studying all aspects of this subject. We often use Mathematica to simplify complicated algebraic manipulations, but avoid its use where possible. These notes are still a work in progress and will be regularly updated - many of the sections are still incomplete.

I would like to thank some of my teachers whose courses I attended at various times and from whom I learnt a lot about dynamical systems. These include William Parry, Peter Walters and Rufus Bowen at the University of Warwick, Dan Newton at the University of Sussex, Michael Sears and Harvey Keynes at the University of the Witwatersrand, and Dan Rudolph at the University of Maryland.

I welcome any comments for improvement and would like to hear about any typographical or mathematical errors.

Geoffrey Goodson, Towson University, 2014.

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Dynamical Systems and Chaos: Spring 2013

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Chapter 1. The Orbits of One-Dimensional Maps.

1.1 Iteration of Functions and Examples of Dynamical Systems.

Dynamical systems is the study of how things change over time. Examples include the growth of populations, the change in the weather, radioactive decay, mixing of liquids and gases such as the ocean currents, motion of the planets, the interest in a bank account. Ideally we would like to study these with continuously varying time (what are called *flows*), but in this text we will simplify matters by only considering discrete changes in time. For example, we might model a population by measuring it daily. Suppose that x_n is the number of members of a population on day n , where x_0 is the initial population. We look for a function $f : \mathbb{R} \rightarrow \mathbb{R}$, (where $\mathbb{R}$ = set of all real numbers), for which

$$x_1 = f(x_0), x_2 = f(x_1) \quad \text{and generally} \quad x_n = f(x_{n-1}), \quad n = 1, 2, \dots$$

This leads to the iteration of functions in the following way:

Definition 1.1.1 Given $x_0 \in \mathbb{R}$, the *orbit* of x_0 under f is the set

$$O(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\},$$

where $f^2(x_0) = f(f(x_0))$, $f^3(x_0) = f(f^2(x_0))$, and continuing indefinitely, so that

$$f^n(x) = f \circ f \circ f \circ \dots \circ f(x); \quad (n\text{-times composition}).$$

Set $x_n = f^n(x_0)$, $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, so that in general

$$x_{n+1} = f^{n+1}(x_0) = f(f^n(x_0)) = f(x_n).$$

More generally, f may be defined on some subinterval I of $\mathbb{R}$, but in order for the iterates of $x \in I$ under f to be defined, we need the range of f to be contained in I , so $f : I \rightarrow I$.

This is what we call the iteration of one-dimensional maps (as opposed to higher dimensional maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n > 1$, which will be studied briefly in a later chapter).

Definition 1.1.2 A (one dimensional) *dynamical system* is a function $f : I \rightarrow I$ where I is some subinterval of $\mathbb{R}$.

Given such a function f , equations of the form $x_{n+1} = f(x_n)$ are examples of *difference equations*. These arise in the types of examples we mentioned above. For

example in biology x_n may represent the number of bacteria in a culture after n hours. There is an obvious correspondence between one-dimensional maps and these difference equations. For example, a difference equation commonly used for calculating square roots:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),$$

corresponds to the function $f(x) = \frac{1}{2}(x + \frac{2}{x})$. If we start by setting $x_0 = 2$ (or in fact any real number), and then find x_1, x_2 etc., we get a sequence which rapidly approaches $\sqrt{2}$ (see page 9 of Sternberg [63]). One of the issues we examine is what exactly is happening here.

Examples of Dynamical Systems 1.1.3

1. The Trigonometric Functions Consider the iterations of the trigonometric functions starting with $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = \sin(x)$. Select $x_0 \in \mathbb{R}$ at random, e.g., $x_0 = 2$ and set $x_{n+1} = \sin(x_n)$, $n = 0, 1, 2, \dots$. Can you guess what happens to x_n as n increases? One way to investigate this type of dynamical system is to enter 2 into our calculator, then repeatedly press the **2nd**, **Answer** and **Sin** keys (you will need to do this many times to get a good idea - it may be easier to use Mathematica, or some similar computer algebra system). Do the same, but replace the sine function with the cosine function. How do we explain what appears to be happening in each case? These are questions that we aim to answer quite soon.

2. Linear Maps Probably the simplest dynamical system (and least interesting from a chaotic dynamical point of view), for population growth arises from the iteration of *linear maps*: maps of the form $f(x) = a \cdot x$. Suppose that x_n = size of a population at time n , with the property

$$x_{n+1} = a \cdot x_n$$

for some constant $a > 0$. This is an example of a *linear model* for the growth of the population.

If the initial population is $x_0 > 0$, then $x_1 = ax_0$, $x_2 = ax_1 = a^2x_0$, and in general $x_n = a^n x_0$ for $n = 0, 1, 2, \dots$. This is the exact solution (or closed form solution) to the difference equation $x_{n+1} = a \cdot x_n$. Clearly $f(x) = ax$ is the corresponding dynamical system. We can use the solution to determine the long term behavior of the population:

x_n is very well behaved since:

(i) if $a > 1$, then $x_n \rightarrow \infty$ as $n \rightarrow \infty$,

- (ii) if $0 < a < 1$ then $x_n \rightarrow 0$ as $n \rightarrow \infty$ (i.e., the population becomes extinct),
- (iii) if $a = 1$, then the population remains unchanged.

3. Affine maps These are functions $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = ax + b$ ($a \neq 0$), for constants a and b . Consider the iterates of such maps:

$$f^2(x) = f(ax + b) = a(ax + b) + b = a^2x + ab + b,$$

$$f^3(x) = a^3x + a^2b + ab + b,$$

$$f^4(x) = a^4x + a^3b + a^2b + ab + b,$$

and generally

$$f^n(x) = a^n x + a^{n-1}b + a^{n-2}b + \cdots + ab + b.$$

Let $x_0 \in \mathbb{R}$ and set $x_n = f^n(x_0)$, then we have

$$\begin{aligned} x_n &= a^n x_0 + (a^{n-1} + a^{n-2} + \cdots + a + 1)b \\ &= a^n x_0 + b \left(\frac{a^n - 1}{a - 1} \right), \quad \text{if } a \neq 1, \end{aligned}$$

or

$$x_n = \left(x_0 + \frac{b}{a - 1} \right) a^n + \frac{b}{1 - a}, \quad \text{if } a \neq 1,$$

is the closed form solution in this case (here we have used the formula for the sum of a finite geometric series:

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1},$$

when $r \neq 1$). If $a = 1$, the solution is $x_n = x_0 + nb$.

We can use these equations to determine the long term behavior of x_n . We see that:

- (i) if $|a| < 1$ then $a^n \rightarrow 0$ as $n \rightarrow \infty$, so that

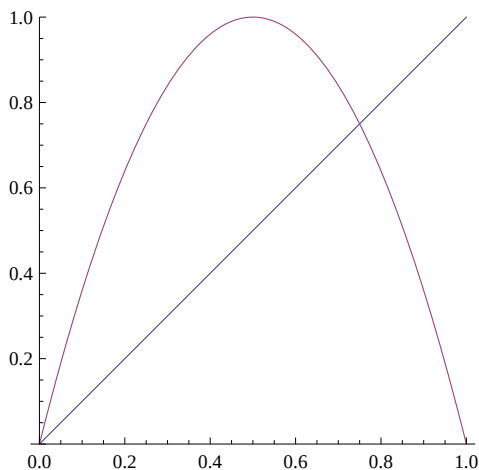
$$\lim_{n \rightarrow \infty} x_n = \frac{b}{1 - a},$$

- (ii) if $a > 1$, then $\lim_{n \rightarrow \infty} x_n = \infty$ for $b, x_0 > 0$,

- (iii) if $a = 1$, then $\lim_{n \rightarrow \infty} x_n = \infty$ if $b > 0$.

The limit won't exist if $a \leq -1$.

4. **The Logistic Map** $L_\mu : \mathbb{R} \rightarrow \mathbb{R}$, $L_\mu(x) = \mu x(1 - x)$ was introduced to model a certain type of population growth (see [41]). Here μ is a real parameter which is fixed. Note that if $0 < \mu \leq 4$ then L_μ is a dynamical system of the interval $[0, 1]$, i.e. $L_\mu : [0, 1] \rightarrow [0, 1]$. For example, when $\mu = 4$, $L_4(x) = 4x(1 - x)$, with $L_4([0, 1]) = [0, 1]$ with graph given below. If $\mu > 4$, then L_μ is no longer a dynamical system of $[0, 1]$ as $L_\mu([0, 1])$ is not a subset of $[0, 1]$.



Recurrence Relations 1.1.4

Many sequences can be defined *recursively* by specifying the first term (or two), and then stating a general rule which specifies how to obtain the n th term from the $(n - 1)$ th term (or other additional terms), and using mathematical induction to see that the sequence is “well defined” for every $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$. For example, $n!$ = n -factorial can be defined in this way by specifying $0! = 1$, and $n! = n \cdot (n - 1)!$, for $n \in \mathbb{Z}^+$. The Fibonacci sequence F_n , can be defined by setting

$$F_0 = 1, \quad F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}, \quad \text{for } n \in \mathbb{Z}^+,$$

so that $F_2 = 2$, $F_3 = 5$ etc.

The orbit of a point $x_0 \in \mathbb{R}$ under a function f is then defined recursively as follows: we are given the starting value x_0 , and we set

$$x_n = f(x_{n-1}), \quad \text{for } n \in \mathbb{Z}^+.$$

The principle of mathematical induction then tells that x_n is defined for every $n \geq 0$, since it is defined for $n = 0$, and assuming it has been defined for $k = n - 1$ then $x_n = f(x_{n-1})$ defines it for $k = n$.

Ideally, given a recursively defined sequence x_n , we would like to have a specific formula for x_n in terms of elementary functions (so called *closed form solution*), but this is often very difficult, or impossible to achieve. In the case of affine maps and certain logistic maps, there is a closed form solution. One can use the former to study problems of the following type:

Example 1.1.5 An amount \$ T is deposited in your bank account at the end of each month. The interest is $r\%$ per period. Find the amount $A(n)$ accumulated at the end of n months (Assume $A(0) = T$).

Answer. $A(n)$ satisfies the difference equation

$$A(n+1) = A(n) + A(n)r + T, \quad \text{where } A(0) = T,$$

or

$$A(n+1) = A(n)(1+r) + T,$$

so we set $x_0 = T$, $a = 1+r$ and $b = T$ in the formula of Example 1.1.3, then the solution is

$$\begin{aligned} A(n) &= (1+r)^n T + T \left(\frac{(1+r)^n - 1}{1+r-1} \right) \\ &= (1+r)^n T + \frac{T}{r} ((1+r)^n - 1). \end{aligned}$$

Remark 1.1.6 It is conjectured that closed form solutions for the difference equation arising from the logistic map are only possible when $\mu = \pm 2, 4$ (see Exercises 1.1 # 3 for the cases where $\mu = 2, 4$, and # 7 for the case where $\mu = -2$ and also [67] for a discussion of this conjecture).

Exercises 1.1

1. If $L_\mu(x) = \mu x(1-x)$ is the logistic map, calculate $L_\mu^2(x)$ and $L_\mu^3(x)$.
2. Use Example 1.1.3 for affine maps to find the solutions to the difference equations:
 - (i) $x_{n+1} - \frac{x_n}{3} = 2$, $x_0 = 2$,

(ii) $x_{n+1} + 3x_n = 4$, $x_0 = -1$.

3. A logistic difference equation is one of the form $x_{n+1} = \mu x_n(1 - x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact solutions to the logistic equations:

(i) $x_{n+1} = 2x_n(1 - x_n)$. Hint: use the substitution $x_n = (1 - y_n)/2$ to transform the equation into a simpler equation that is easily solved.

(ii) $x_{n+1} = 4x_n(1 - x_n)$. Hint: set $x_n = \sin^2(\theta_n)$ and simplify to get an equation that is easily solved.

4. You borrow \$ P at $r\%$ per annum, and pay off \$ M at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100,000$, $M = 1000$ and $r = 4$, after how long will the loan be paid off?

5. If $T_\mu(x) = \begin{cases} \mu x & \text{if } 0 \leq x < 1/2 \\ \mu(1 - x) & \text{if } 1/2 \leq x < 1 \end{cases}$, show that T_μ is a dynamical system of $[0, 1]$ for $\mu \in (0, 2]$

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined below. For each of the intervals given, determine whether f can be considered as a dynamical system $f : I \rightarrow I$:

(a) $f(x) = x^3 - 3x$,

(i) $I = [-1, 1]$, (ii) $I = [-2, 2]$.

(b) $f(x) = 2x^3 - 6x$,

(i) $I = [-1, 1]$, (ii) $I = [-\sqrt{7/2}, \sqrt{7/2}]$, (iii) $I = [-4, 4]$.

7. For the following functions, find $f^2(x)$, $f^3(x)$ and a general formula for $f^n(x)$:

(i) $f(x) = x^2$, (ii) $f(x) = |x + 1|$, (iii) $f(x) = \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x < 1. \end{cases}$

8. Use mathematical induction to show that if $f(x) = \frac{2}{x+1}$, then

$$f^n(x) = \frac{2^n(x+2) + (-1)^n(2x-2)}{2^n(x+2) - (-1)^n(x-1)}.$$

9*. Show that a closed form solution to the logistic difference equation when $\mu = -2$ is given by

$$x_n = \frac{1}{2} \left[1 - f \left[r^n f^{-1}(1 - 2x_0) \right] \right], \quad \text{where } r = -2 \quad \text{and} \quad f(\theta) = 2 \cos \left(\frac{\pi - \sqrt{3}\theta}{3} \right).$$

(Hint: Set $x_n = \frac{1 - f(\theta_n)}{2}$ and use steps similar to 3(ii)).

1.2 Newton's Method and Fixed Points

Given a differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, Newton's method often allows us to find good approximations to zeros of $f(x)$, i.e., approximate solutions to the equation $f(x) = 0$. The idea is to start with a first approximation x_0 and look at the tangent line to $f(x)$ at the point $(x_0, f(x_0))$. Suppose this line intersects the x -axis at x_1 , then we can check that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad \text{if } f'(x_0) \neq 0.$$

For reasons that we will make clear, it is often the case that x_1 is a better approximation to the zero $x = c$ than x_0 was. In other words, Newton's method is an algorithm for finding approximations to zeros of a function $f(x)$. The algorithm gives rise to a difference equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

where x_0 is a first approximation to a zero of $f(x)$. The corresponding real function is

$$N_f(x) = x - \frac{f(x)}{f'(x)}, \quad (\text{the Newton function}).$$

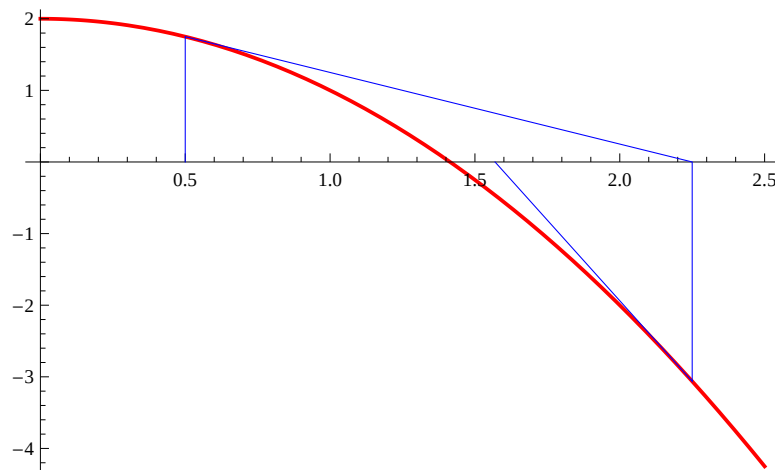
Note that if $f(x) = x^2 - a$, then $f'(x) = 2x$ and

$$N_f(x) = x - \frac{x^2 - a}{2x} = \frac{1}{2} \left(x + \frac{a}{x} \right),$$

so that

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right),$$

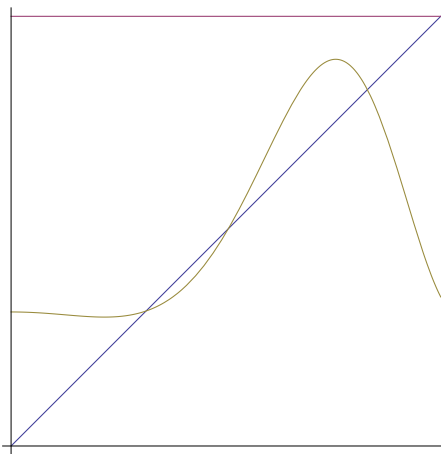
the difference equation we mentioned in Section 1.1 that is used for approximating $\sqrt{2}$ when $a = 2$.



The first two approximations for $f(x) = 2 - x^2$, starting with $x_0 = .5$.

Note that we are looking for where $f(x) = 0$, and this happens if and only if $N_f(x) = x$.

Definition 1.2.1 For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, a point $c \in \mathbb{R}$ for which $f(c) = c$ is called a *fixed point* of f . It is a point where the graph of $f(x)$ intersects the line $y = x$. We denote the set of fixed points of f by $\text{Fix}(f)$.



Fixed points occur where the graph of $f(x)$ intersects the line $y = x$.

Example 1.2.2 Suppose that $f(x) = x^2$, then $x^2 = x$ gives $x(x - 1) = 0$, so has fixed points $c = 0$ and $c = 1$, so $\text{Fix}(f) = \{0, 1\}$. If $f(x) = x^3 - x$, then $x^3 - x = x$ gives $x(x^2 - 2) = 0$, so the fixed points are $c = 0$ and $c = \pm\sqrt{2}$, $\text{Fix}(f) = \{0, \pm\sqrt{2}\}$.

The logistic map $f(x) = 4x(1 - x) = 4x - 4x^2$, $0 \leq x \leq 1$ has the properties: $f(0) = 0$, (a fixed point), $f(1) = 0$, a maximum at $x = 1/2$ (with $f(1/2) = 1$). Solving $f(x) = x$ gives $4x - 4x^2 = x$, so $4x^2 = 3x$, so the fixed points are $c = 0$ and $c = 3/4$.

This map has what we call *eventual fixed points*: $f(1) = 0$ and $f(0) = 0$, so we say that $c = 1$ is eventually fixed. Also $f(1/4) = 3/4$, so $c = 1/4$ is eventually fixed, as is $c = (2 + \sqrt{3})/4$.

Definition 1.2.3 $x^* \in \mathbb{R}$ is an *eventual fixed point* of $f(x)$ if there exists a fixed point c of $f(x)$ and $r \in \mathbb{Z}^+$ satisfying $f^r(x^*) = c$, but $f^s(x^*) \neq c$ for $0 < s < r$.

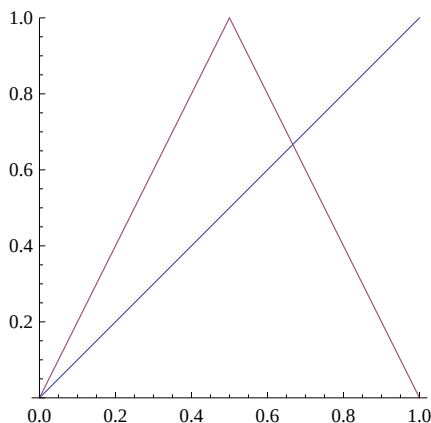
Example 1.2.4 The Tent Map

Define a function $T : [0, 1] \rightarrow [0, 1]$ by

$$T(x) = \begin{cases} 2x & : 0 \leq x \leq 1/2 \\ 2(1 - x) & : 1/2 < x \leq 1 \end{cases}$$

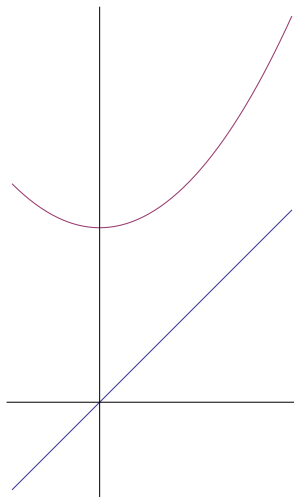
$$= 1 - 2|x - 1/2|.$$

$T(x)$ is called the *tent map*, it has fixed points: $T(0) = 0$ and $T(2/3) = 2/3$. Since $T(1/4) = 1/2$, $T(1/2) = 1$ and $T(1) = 0$, $1/4$, $1/2$ and 1 are eventually fixed. It is not difficult to see that there are many other eventually fixed points (infinitely many).



Note that some maps do not have fixed points:

Example 1.2.5 $f(x) = x^2 + 1$ has no fixed points.

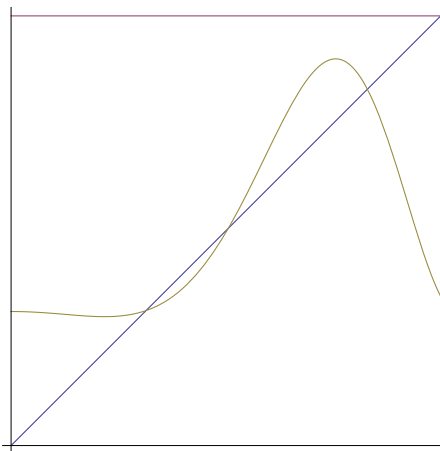


$f(x) = x^2 + 1$ does not intersect the line $y = x$.

Question 1.2.6 For what values of $a \in \mathbb{R}$ does $Q_a(x) = x^2 + a$ have fixed points? We see below that certain functions always have fixed points:

Theorem 1.2.7 Let $f : I \rightarrow I$ be a continuous function, where $I = [a, b]$, $a < b$ is a bounded interval. Then $f(x)$ has a fixed point $c \in I$.

Proof. Set $g(x) = f(x) - x$. We may assume that $f(a) \neq a$ and $f(b) \neq b$, so that $f(a) > a$ and $f(b) < b$.



The graph of $f(x)$ always intersects the line $y = x$.

It follows that

$$g(a) = f(a) - a > 0, \quad \text{and} \quad g(b) = f(b) - b < 0,$$

so that $g(x)$ is a continuous function which is positive at a and negative at b , so by the Intermediate Value Theorem (IVT), there exists $c \in (a, b)$ (open interval) with $g(c) = 0$, i.e., $f(c) = c$, so c is a fixed point of $f(x)$. $\square$

Remark 1.2.8 The above is an example of an existence theorem, it says nothing about how to find the fixed point, where it is or how many there are. It tells us that if $f(x)$ is a continuous function on an interval I with $f(I) \subseteq I$, then $f(x)$ has a fixed point in I . It is also true that if instead $f(I) \supseteq I$ then $f(x)$ has a fixed point in I as we see now:

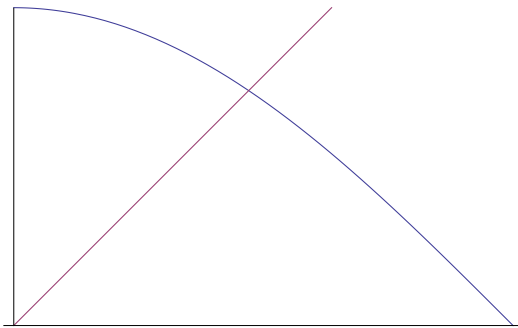
Theorem 1.2.9 *Let $f : I \rightarrow \mathbb{R}$ ($I = [a, b], a < b$) be a continuous function with $f(I) \supseteq I$, then $f(x)$ has a fixed point in I .*

Proof. As before, set $g(x) = f(x) - x$, then there exists $c_1 \in (a, b)$ with $f(c_1) < c_1$ (in fact $f(c_1) < a < c_1$). Also there is $c_2 \in (a, b)$ with $f(c_2) > c_2$.

Then $g(c_1) < 0$ and $g(c_2) > 0$ and since $g(x)$ is a continuous function, it follows by the Intermediate Value Theorem that there exists $c \in I$, ($c_1 < c < c_2$ or $c_2 < c < c_1$), with $g(c) = 0$; $f(c) = c$. $\square$

Remark 1.2.10 It is often difficult to find fixed points explicitly:

Example 1.2.11 Set $f(x) = \cos x$. If $f(c) = c$, then $\cos c = c$. It is possible to find an approximation to the fixed point $c = .739085\dots$ using (for example) Newton's method.



$f(x) = \cos x$ has a fixed point in $[0, \pi/2]$.

Exercises 1.2

1. Find all fixed points and eventual fixed points of the map $f(x) = |x - 1|$.
2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that for some $n \in \mathbb{Z}^+$, the n th iterate of f has a unique fixed point c (i.e., $f^n(c) = c$ and c is unique). Then show that c is a fixed point of f .
3. Use the **Mathematica** command **NestList** (see below) to find how the following functions behave when the given point is iterated (comment on what appears to be happening in each case):
 - (i) $L_1(x) = x(1 - x)$, with starting point $x_0 = .75$.
 - (ii) $L_2(x) = 2x(1 - x)$, with starting point $x_0 = .1$.
 - (iii) $L_3(x) = 3x(1 - x)$, with starting point $x_0 = .2$.
 - (iv) $L_{3.2}(x) = 3.2x(1 - x)$, with starting point $x_0 = .95$.
 - (v) $f(x) = \sin(x)$, with starting point $x_0 = 9.5$.
 - (vi) $g(x) = \cos(x)$, with starting point $x_0 = -15.3$.
4. Show that the logistic map $L_\mu(x) = \mu x(1 - x)$, $x \in [0, 1]$, for $0 < \mu < 4$, the fixed point $x = 0$ has no eventual fixed points other than $x = 1$.
 Show that for $1 < \mu \leq 2$, the fixed point $x = 1 - 1/\mu$ only has $x = 1/\mu$ as an eventual fixed point, but this is not the case for $2 < \mu < 3$.

*To use **Mathematica**, first define your function, say $f(x) = \sin(x)$ in an input cell using the syntax*

```
f[x_] := Sin[x].
```

*Execute the cell by doing **Shift** and **Enter** simultaneously. The command*

```
NestList[f, 9.5, 100]
```

when executed in a new input cell will give 100 iterations of $\sin(x)$ with starting value $x_0 = 9.5$. If you want to graph $f(x)$ on the interval $[a, b]$, use the command:

```
Plot[f[x], { x, a, b } ].
```

5. (a) Let $f(x) = (1 + x)^{-1}$. Find the fixed points of f and show that there are no points c with $f^2(c) = c$ and $f(c) \neq c$ (period 2-points). Note that $f(-1)$ is not defined, but points that get mapped to -1 belong to the interval $[-2, -1)$ and are of the form

$$\nu_n = -\frac{F_{n+1}}{F_n}, \quad n \geq 1,$$

where $\{F_n\}$, $n \geq 0$, is the Fibonacci sequence (see 1.1.4). Note that $\nu_n \rightarrow -r$ as $n \rightarrow \infty$, where $-r$ is the negative fixed point of f (see [12] for more details).

(b) If $x_0 \in (0, 1]$, set $x_n = \frac{F_{n-1}x_0 + F_n}{F_nx_0 + F_{n+1}}$. Use mathematical induction to show that

$$x_{n+1} = f(x_n) = \frac{F_nx_0 + F_{n+1}}{F_{n+1}x_0 + F_{n+2}}.$$

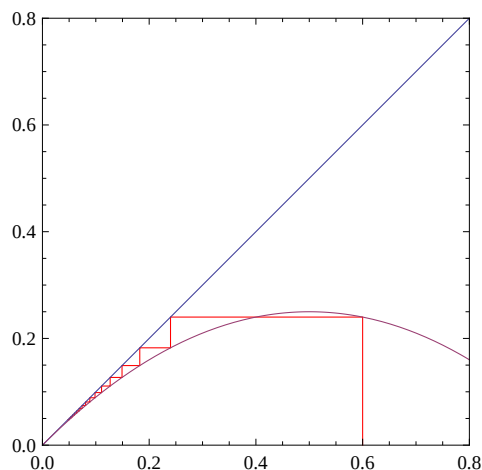
Deduce that as $n \rightarrow \infty$, $x_n \rightarrow 1/r$, the positive fixed point of f .

1.3 Graphical Iteration

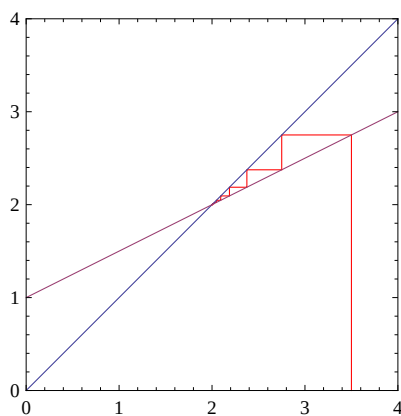
It is possible to follow the iterates of a function $f(x)$ at a point x_0 using *graphical iteration* (sometimes called *web diagrams*).

We start at x_0 on the x -axis and draw a line vertically to the function. We then move horizontally to the line $y = x$, then vertically to the function and continue in this way. Notice that in some examples the iterations converges to a fixed point. In others it goes off to ∞ , whilst in others still, it oscillates between two points indefinitely.

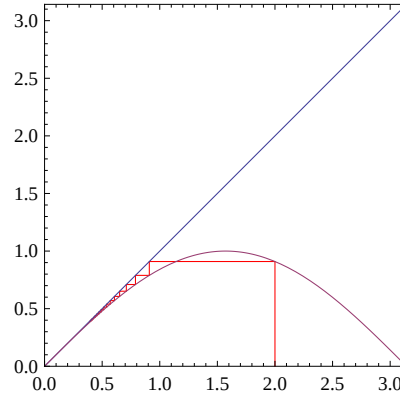
Example 1.3.1 $f(x) = x(1 - x)$. In this example, an examination of graphical iteration seems to suggest that the orbits of any point in $[0, 1]$ approach the fixed point $x = 0$. This is an example of an *attracting* fixed point.



Example 1.3.2 Let $f(x) = x/2 + 1$. This is an affine transformation with $a = 1/2$, $b = 1$. According to what we saw in Example 1.1.3, the iterates should converge to $\frac{b}{1-a} = 2$ since $|a| < 1$. What is actually happening is that $c = 2$ is an attracting fixed point of $f(x)$ with the property that it attracts all members of $\mathbb{R}$ (said to be *globally attracting*).

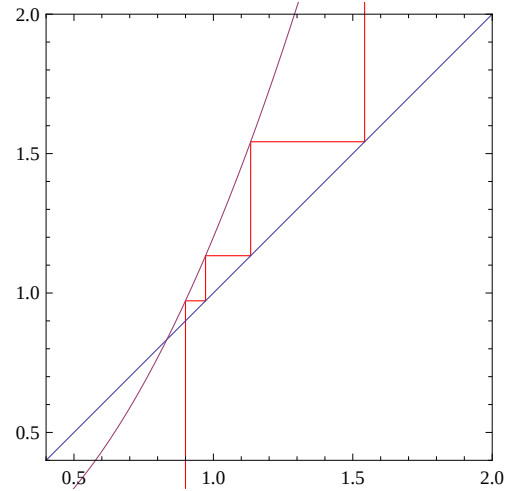
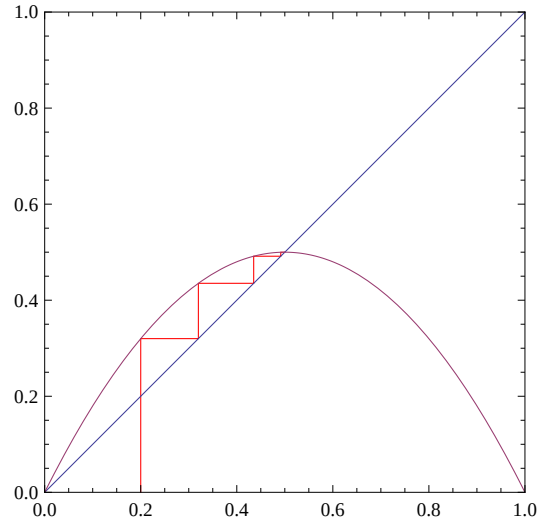


Example 1.3.3 We see from the graph of $f(x) = \sin x$ that $c = 0$ seems to be an attracting fixed point. We shall show later that it is actually globally attracting.



We see two basic situations arising (together with some variations of these). In particular, we notice that if the sequence $x_n = f^n(x_0)$ converges to some point c as $n \rightarrow \infty$, then c is a fixed point.

- (i) Stable orbit, where the web diagram approaches a fixed point.
- (ii) Unstable orbit, where the web diagram moves away from a fixed point.



Proposition 1.3.4 *If $y = f(x)$ is a continuous function and $\lim_{n \rightarrow \infty} f^n(x_0) = c$, then $f(c) = c$ (i.e., if the orbit converges to a point c , then c is a fixed point of f).*

Proof. We see that

$$\lim_{n \rightarrow \infty} f^n(x_0) = c \Rightarrow f(\lim_{n \rightarrow \infty} f^n(x_0)) = f(c),$$

and since f is continuous,

$$\lim_{n \rightarrow \infty} f^{n+1}(x_0) = f(c).$$

But clearly $c = \lim_{n \rightarrow \infty} f^{n+1}(x_0)$, so that $c = f(c)$ by the uniqueness of limit. $\square$

1.4 Attractors and Repellers

Suppose that $f : X \rightarrow X$ is a function (where we assume that X is some subset of $\mathbb{R}$ such as an interval or possibly $\mathbb{R}$ itself). Let $c \in X$ be a fixed point of f . We define the notions on attracting fixed point and repelling fixed that we looked at graphically in the last section.

Definition 1.4.1 Let $f : X \rightarrow X$ with $f(c) = c$.

(i) c is a *stable fixed point* if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in X$ and $|x - c| < \delta$, then $|f^n(x) - c| < \epsilon$ for all $n \in \mathbb{Z}^+$.

If this does not hold, c will be called *unstable*.

(ii) c is said to be *attracting* if there is a real number $\eta > 0$ such that

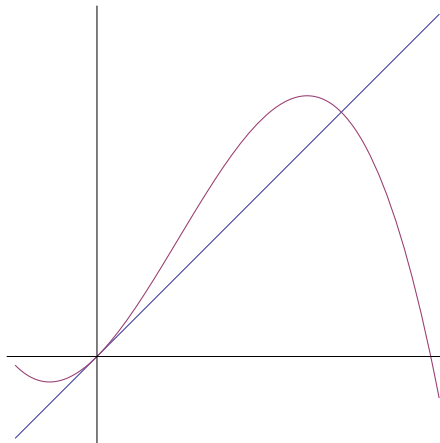
$$|x - c| < \eta \Rightarrow \lim_{n \rightarrow \infty} f^n(x) = c.$$

(iii) c is *asymptotically stable* if it is both stable and attracting.

Remark 1.4.2 We will show that a fixed point c of $f : X \rightarrow X$ (suitably differentiable) with the property $|f'(c)| < 1$ is an asymptotically stable fixed point. If c is an unstable fixed point, we can find $\epsilon > 0$ and x arbitrarily close to c such that some iterate of x , say $f^n(x)$, will be greater than distance ϵ from c . This is the case when $|f'(c)| > 1$ as the next theorem shows. We call c a *repeller* (c is a *repelling fixed point*), since the iterates move away from the fixed point (c is unstable). We will also see that a fixed point can be stable without being attracting, and that it can be attracting without being stable.

Definition 1.4.3 A fixed point c of f is *hyperbolic* if $|f'(c)| \neq 1$. If $|f'(c)| = 1$ it is *non-hyperbolic*.

Thus if c is a non-hyperbolic fixed point, then $f'(c) = 1$ or $f'(c) = -1$, so the graph of $f(x)$ either meets the line $y = x$ tangentially, or at 90° :



$f(x) = -2x^3 + 2x^2 + x$ has both types of non-hyperbolic fixed points.

Hyperbolic fixed points have the following properties:

Theorem 1.4.4 *Let $f : X \rightarrow X$ be a differentiable function with continuous first derivative (we say that f is of class C^1).*

(i) *If a is a fixed point for $f(x)$ with $|f'(a)| < 1$, then a is asymptotically stable. The iterates of points close to a , converge to a geometrically (i.e., there is a constant $0 < \lambda < 1$ for which $|f^n(x) - a| < \lambda^n |x - a|$ for all $n \in \mathbb{Z}^+$ and for all $x \in X$ sufficiently close to a).*

(ii) *If a is a fixed point for $f(x)$ for which $|f'(a)| > 1$, then a is a repelling fixed point for f .*

Proof. (i) We may assume that X is an open interval with $a \in X$. Suppose $|f'(a)| < \lambda < 1$ for some $\lambda > 0$, then using the continuity of $f'(x)$, there exists an open interval I with $|f'(x)| < \lambda < 1$ for all $x \in I$.

By the Mean Value Theorem, if $x \in I$ there exists $c \in I$, lying between x and a , satisfying

$$f'(c) = \frac{f(x) - f(a)}{x - a},$$

so that

$$|f(x) - a| = |f'(c)| |x - a| < \lambda |x - a|,$$

(i.e., $f(x)$ is closer to a than x was).

Repeating this argument with $f(x)$ replacing x gives

$$|f^2(x) - a| < \lambda^2 |x - a|, \dots, |f^n(x) - a| < \lambda^n |x - a|.$$

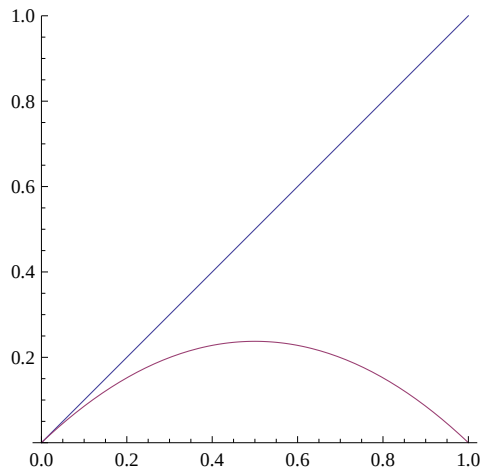
Since $\lambda^n \rightarrow 0$ as $n \rightarrow \infty$, it follows that $f^n(x) \rightarrow a$ as $n \rightarrow \infty$.

The proof of (ii) is similar. □

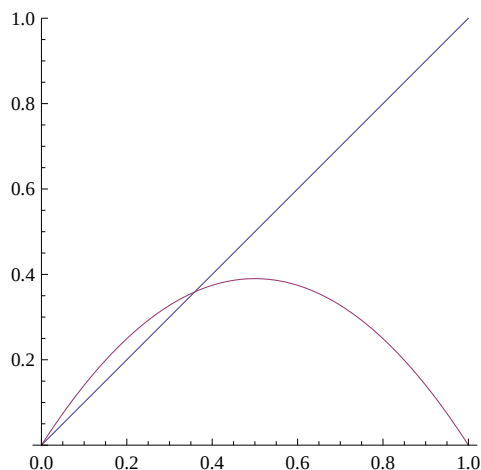
Example 1.4.5 Denote the Logistic map by $L_\mu(x) = \mu x(1 - x) = \mu x - \mu x^2$. We are interested in this map for $0 \leq x \leq 1$ and $0 < \mu \leq 4$, since for these parameter values μ , L_μ is a function which maps the interval $[0, 1]$ into the interval $[0, 1]$, so it necessarily has a fixed point and iterates stay inside $[0, 1]$ (if $\mu > 4$, $L_\mu(x) > 1$ for some values of x in $[0, 1]$ and further iterates will go to $-\infty$).

Solving $L_\mu(x) = x$ gives $x = 0$ or $x = 1 - 1/\mu$. If $0 < \mu \leq 1$, then $1 - 1/\mu \leq 0$, so $c = 0$ is the only fixed point in $[0, 1]$ in this case.

For $0 < \mu \leq 1$, $L'_\mu(x) = \mu - 2\mu x$, so $L'_\mu(0) = \mu$ and 0 is an attracting fixed point for $0 < \mu < 1$.



If $\mu > 1$, then 0 and $1 - 1/\mu$ are both fixed points in $[0, 1]$, but now 0 is repelling.



Also,

$$L'_\mu(1 - 1/\mu) = \mu - 2\mu(1 - 1/\mu) = 2 - \mu,$$

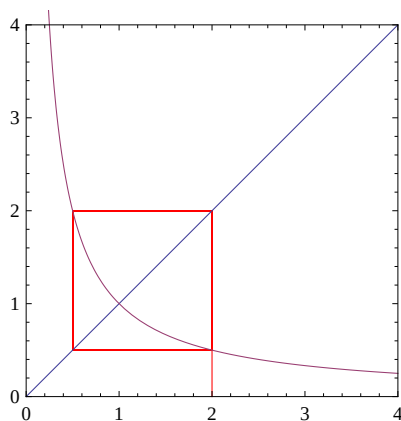
so that

$$|L'_\mu(1 - 1/\mu)| = |2 - \mu| < 1 \quad \text{iff} \quad 1 < \mu < 3.$$

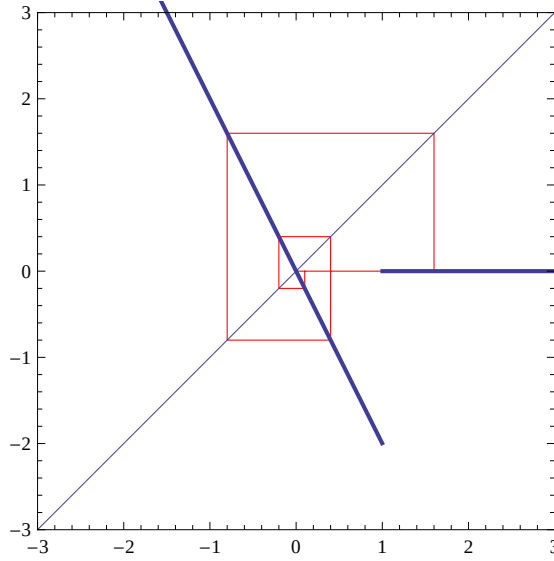
In this case $c = 1 - 1/\mu$ is an attracting fixed point, and repelling for $\mu > 3$. Note that $L'_1(0) = 1$ and $L'_3(2/3) = 1$ so we have non-hyperbolic fixed points.

We will examine in Chapter 2 what happens in the cases where $\mu = 1$ and $\mu = 3$.

Example 1.4.6 Let $f(x) = 1/x$. The two fixed points $x = \pm 1$ are hyperbolic since $f'(x) = -1/x^2$, so $f'(\pm 1) = -1$. However, they are stable but not attracting since $f^2(x) = f(1/x) = x$, and we see points close to ± 1 neither move closer nor further apart.



Example 1.4.7 Consider instead the function $f_a(x) = \begin{cases} -2x & \text{if } x < a \\ 0 & \text{if } x \geq a \end{cases}$, where a is any positive real number. Then $c = 0$ is an unstable (repelling) fixed point of f_a which is attracting i.e., $\lim_{n \rightarrow \infty} f^n(x) = 0$ for all $x \in \mathbb{R}$. A fixed point having this latter property is said to be *globally attracting*. Note that the function f_a is not continuous at $x = a$. It has been shown by Sedaghat [58] that a continuous mapping of the real line cannot have an unstable fixed point that is globally attracting.



The map f_1 (in bold) where $x = 0$ is an attracting but unstable fixed point.

Example 1.4.8 Newton's Method Revisited.

Suppose that $f : I \rightarrow I$ is a function whose zero c is to be approximated using Newton's method. The Newton function is $N_f(x) = x - f(x)/f'(x)$, where we are assuming $f'(c) \neq 0$. Notice that since $f(c) = 0$, $N_f(c) = c$, i.e., c is a fixed point of N_f . Consider $N'_f(c)$:

$$N'_f(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

so that

$$N'_f(c) = \frac{f(c)f''(c)}{[f'(c)]^2} = 0,$$

since $f(c) = 0$.

It follows that $|N'_f(c)| = 0 < 1$, so that c is an attracting fixed point for N_f , and in particular

$$\lim_{n \rightarrow \infty} N_f^n(x_0) = c,$$

provided x_0 , the first approximation to c , is sufficiently close to c .

Definition 1.4.9 A fixed point c for $f(x)$ is called a *super-attracting fixed-point* if $f'(c) = 0$. This gives a very fast convergence to the fixed-point for points nearby.

Remark 1.4.10 Suppose that $f'(c) = 0$, then $N_f(x)$ is not defined at $x = c$, since the quotient $f(x)/f'(x)$ is not defined there. Suppose we can write $f(x) = (x - c)^k h(x)$ where $h(c) \neq 0$ and $k \in \mathbb{Z}^+$ (for example if $f(x)$ is a polynomial with a multiple root), then we have

$$\frac{f(x)}{f'(x)} = \frac{(x - c)^k h(x)}{k(x - c)^{k-1} h(x) + (x - c)^k h'(x)} = \frac{(x - c)h(x)}{kh(x) + (x - c)h'(x)} = 0,$$

when $x = c$, so that $N_f(x) = c$ has a removable discontinuity at $x = c$ (removed by setting $N_f(c) = c$). Then we can find $N'_f(x)$ and show that $N'_f(x)$ has a removable discontinuity at $x = c$, which can again be removed by setting $N'_f(c) = (k - 1)/k$, so that $|N'_f(c)| < 1$ (see the exercises).

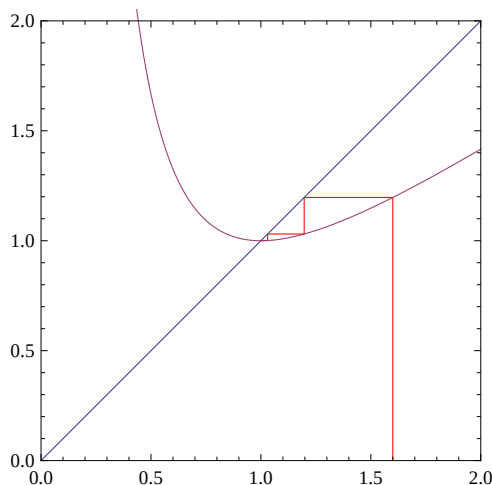
We summarize the above with a theorem:

Theorem 1.4.11 For a differentiable function $f : I \rightarrow I$, the zero c of $f(x)$ is a super-attracting fixed point of the Newton function N_f , if and only if $f'(c) \neq 0$.

Example 1.4.12 Suppose $f(x) = x^3 - 1$, then $f(1) = 0$ and

$$N_f(x) = x - f(x)/f'(x) = x - \left(\frac{x^3 - 1}{3x^2} \right) = \frac{2x}{3} + \frac{1}{3x^2},$$

so $N_f(1) = 1$ and $N'_f(x) = \frac{2}{3} - \frac{2}{3x^3}$, so that $N'_f(1) = 0$. We see from graphical iteration (next section), very fast convergence to the fixed point of N_f .



Very fast convergence to the fixed point.

Exercises 1.4

1. Find the fixed points and determine their stability for the function $f(x) = 4 - \frac{3}{x}$.
2. For the family of quadratic maps $Q_c(x) = x^2 + c$, $x \in [0, 1]$, use the Mathematica program `webPlot` below to give graphical iteration for the values shown (use 20 iterations):
 - (i) $c = 1/2$, starting point $x_0 = 1$, (ii) $c = 1/4$, starting point $x_0 = .1$, (iii) $c = 1/8$, starting point $x_0 = .7$.

Mathematica Program: Use as in:

```
webPlot[Cos[x], {x, 0, Pi}, {.1, 20}, AspectRatio->Automatic]

webPlot[g_, {x_, xmin_, xmax_}, {a1_, n_}, opts_] :=
Module[{seq, r, pts, web, graph}, r[t_] := N[g /. x -> t];
  seq = NestList[r, a1, n];      pts = Flatten[
Table[{{seq[[i]], seq[[i + 1]]}, {seq[[i + 1]], seq[[i + 1]]}}, {i, 1, n}], 1];
  web = Graphics[{Hue[0], Line[PrependTo[pts, {seq[[1]], 0}]]];
  graph = Plot[{x, r[x]}, {x, xmin, xmax},
    DisplayFunction -> Identity]; Print["last iterate =", Last[seq]];
  Show[web, graph, opts, Frame -> True,
```

PlotRange -> {{xmin, xmax}, {xmin, xmax}}]]

3. Let $S_\mu(x) = \mu \sin(x)$, $0 \leq \mu \leq 2\pi$, $0 \leq \mu \leq \pi$ and $C_\mu(x) = \mu \cos(x)$, $-\pi \leq x \leq \pi$ and $0 \leq \mu \leq \pi$.

(a) Show that S_μ has a super-attracting fixed point at $x = \pi/2$, when $\mu = \pi/2$.

(b) Find the corresponding values for C_μ having a super-attracting fixed point.

4. Show that the map $f(x) = \frac{2}{x+1}$ has no periodic points of period $n > 1$ (Hint: Use the closed formula in Exercise 1.1.6).

5. Show that if $f(x) = x + 1/x$ and $x > 0$, then $R^n(x) \rightarrow \infty$ as $n \rightarrow \infty$. (Hint: R has no fixed points in $\mathbb{R}$ and it is continuous with $0 < R(x) < R^2(x) < \dots$, so suppose the limit exists and obtain a contradiction).

6. Let N_f be the Newton function of the map $f(x) = x^2 + 1$. Clearly there are no fixed points of the the Newton function as there are no zeros of f . Show that there are points c where $N_f^2(c) = c$ (called *period 2-points* of N_f).

7. (a) Suppose that $f(c) = f'(c) = 0$ and $f''(c) \neq 0$. If $f''(x)$ is continuous at $x = c$, show that the Newton function $N_f(x)$ has a removable discontinuity at $x = c$ (hint: apply L'Hopital's rule to N_f at $x = c$).

(b) If in addition, $f'''(x)$ is continuous at $x = c$ with $f'''(c) \neq 0$, show that $N'_f(c) = 1/2$, so that $x = c$ is not a super-attracting fixed point in this case.

(c) Check the above for the function $f(x) = x^3 - x^2$ with $c = 0$.

8. Continue the argument in Remark 1.4.11 (generalizing the last exercise), to show that the derivative of the Newton function N_f has a removable discontinuity at $x = c$, which can be removed by setting $N'_f(c) = (k - 1)/k$.

9. Let f be a twice differentiable function with $f(c) = 0$. Show that if we find the Newton function of $g(x) = f(x)/f'(x)$, then $x = c$ will be a super-attracting fixed point for N_g , even if $f'(c) = 0$ (this is called Halley's method).

1.5 Non-hyperbolic Fixed Points.

Example 1.5.1 We have seen that if $f : X \rightarrow X$ (where usually X is some subinterval of $\mathbb{R}$) and $a \in X$ with $f(a) = a$, $|f'(a)| < 1$, with suitable differentiability conditions, then a is a stable fixed point for f . This term is used because points close to a will approach a under iteration. If we look at a function like $f(x) = \sin x$, we see that $c = 0$ is a fixed point but in this case $|f'(0)| = 1$, so it is a non-hyperbolic fixed point. However, graphical iteration suggests that the basin of attraction of f is all of $\mathbb{R}$, so $c = 0$ is a stable fixed point. Before considering non-hyperbolic fixed points in more detail, let us prove the last statement analytically:

Proposition 1.5.2 *The fixed point $c = 0$ of $f(x) = \sin x$ is globally attracting.*

Proof. First notice that $c = 0$ is the only fixed point of $\sin x$. This is clear for if $\sin x = x$ then we cannot have $|x| > 1$ since $|\sin x| \leq 1$ for all x . If $0 < x \leq 1$, the Mean Value Theorem implies there exists $c \in (0, x)$ with

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin x}{x},$$

so that

$$\sin x = x \cos c < x,$$

since $|\cos c| < 1$ for $c \in (0, 1)$. Similarly if $-1 \leq x < 0$.

To show that $f(x)$ is globally attracting, let $x \in \mathbb{R}$. We may assume that $-1 \leq x \leq 1$, since this will be the case after the first iteration.

Suppose that $0 < x \leq 1$, then we note that $0 < f'(x) < 1$ on this interval. By the Mean Value Theorem, there exists $c \in (0, x)$ with

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad \text{or} \quad 0 < f(x) = f'(c)x < x.$$

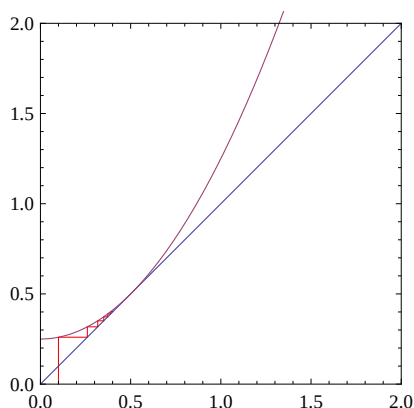
Continuing in this way, we see that

$$0 < f^n(x) < f^{n-1}(x) < \dots < f(x) < x,$$

so we have a decreasing sequence $x_n = f^n(x)$ bounded below by 0. It follows that this sequence converges. But Proposition 1.4.5 implies that if this sequence converges, it

must converge to a fixed point. $c = 0$ being the only fixed point gives $f^n(x) \rightarrow 0$ as $n \rightarrow \infty$. A similar argument can be used for $-1 \leq x < 0$. $\square$

Example 1.5.3 It is possible for the fixed point to be unstable, but to have a one-sided stability (to be *semi-stable*). For example, consider $f(x) = x^2 + 1/4$ which has the single (non-hyperbolic) fixed point $c = 1/2$. This fixed point is stable from the left, but unstable on the right.

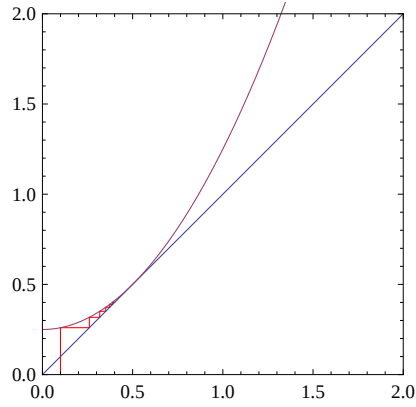


In the following we give some criteria for non-hyperbolic fixed points to be asymptotically stable/unstable etc. It also gives a criteria for semi-stability.

Theorem 1.5.4 Let c be a non-hyperbolic fixed point of $f(x)$ with $f'(c) = 1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at $x = c$, then:

- (i) if $f''(c) \neq 0$, the c is semi-stable,
- (ii) if $f''(c) = 0$ and $f'''(c) > 0$, then c is unstable,
- (iii) If $f''(c) = 0$ and $f'''(c) < 0$, then c is asymptotically stable.

Proof. (i) If $f'(c) = 1$ then $f(x)$ is tangential to $y = x$ at $x = c$. Suppose that $f''(c) > 0$, then $f(x)$ is concave up at $x = c$ and the picture must look like the following:



We see this gives stability on the left and instability on the right.

More formally, since the derivatives are continuous, and $f''(c) > 0$, this will be true in some small interval $(c - \delta, c + \delta)$ surrounding c . In particular, the derivative function $f'(x)$ must be increasing on that interval, so that since $f'(c) = 1$ then

$$f'(x) < 1 \quad \text{for all } x \in (c - \delta, c), \quad \text{and} \quad f'(x) > 1 \quad \text{for all } x \in (c, c + \delta),$$

for some $\delta > 0$. Also, from the continuity of $f'(x)$, we can assume that $f'(x) > 0$ in this interval.

Now by the Mean Value Theorem applied to the interval $[x, c] \subset (c - \delta, c]$, there exists $q \in (x, c)$ with

$$f'(q) = \frac{f(x) - f(c)}{x - c},$$

Now since $0 < f'(q) < 1$ and $c > x$, we have

$$0 < \frac{f(x) - f(c)}{x - c} < 1,$$

or

$$x < f(x) < c.$$

Repeating this argument, we see that the sequence $f^n(x)$ is increasing and bounded above by c , so must converge to a fixed point. There can be no other fixed point (say $d \neq c$), in this interval as the Mean Value Theorem would give $f'(q_1) = 1$ for some $q_1 \in (x, c)$, a contradiction. Consequently we see that c is stable on the left.

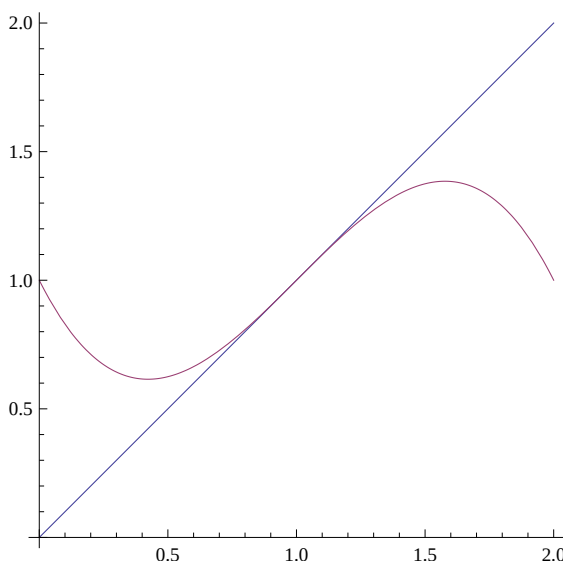
On the other hand, if $[c, x] \subset (c, c + \delta)$, then applying the Mean Value Theorem as above gives

$$f'(q) = \frac{f(x) - f(c)}{x - c} > 1, \quad \text{so} \quad f(x) > x > c,$$

since $x - c > 0$. This tells us that the point moves away from c under iteration, so the fixed point is unstable on the right. Similar considerations can be used when $f'''(c) < 0$ and the graph is concave down at $x = c$.

(ii) is similar to (iii), so is omitted.

(iii) In this case $f'''(c) < 0$, $f''(c) = 0$ and $f'(c) = 1$. We will show that we have a point of inflection at $x = c$ as in the following picture:



By the second derivative test, $f'(x)$ has a local maximum at $x = c$ (the continuous function $f'(x)$ is concave down). It follows that

$$f'(x) < 1 \quad \text{for all } x \in (c - \delta, c + \delta), \quad x \neq c,$$

for some $\delta > 0$ ($f''(x) > 0$ for $x \in (c - \delta, c)$, so $f'(x)$ is increasing there, and $f''(x) < 0$ for $x \in (c, c + \delta)$, so $f'(x)$ is decreasing there). In particular $f'(x) \neq 1$ if $x \neq c$.

Now use an argument similar to that of (i) above to deduce the result. $\square$

Example 1.5.5 Returning to the function $f(x) = \sin x$, we see that $f'(0) = 1$, $f''(0) = 0$ and $f'''(0) = -1$, so the conditions of Theorem 1.5.4 (iii) are satisfied and we conclude that $x = 0$ is an asymptotically stable fixed point.

If $f(x) = \tan x$, then $f'(0) = 1$ and we can check that Theorem 1.5.4 (ii) holds so that the fixed point $x = 0$ is unstable. For $f(x) = x^2 + 1/4$, with $f(1/2) = 1/2$, $f'(1/2) = 1$, we can apply Theorem 1.5.4 (i).

How do we treat the case where $f'(c) = -1$ at the fixed point? We use:

Definition 1.5.6 The *Schwarzian derivative* $Sf(x)$ of $f(x)$ is the function

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2,$$

so that

$$Sf(x) = -f'''(x) - \frac{3}{2}[f''(x)]^2, \quad \text{when } f'(x) = -1.$$

Theorem 1.5.7 Suppose that c is a fixed point for $f(x)$ and $f'(c) = -1$. If $f'(x)$, $f''(x)$ and $f'''(x)$ are continuous at $x = c$ then:

- (i) if $Sf(c) < 0$, then $x = c$ is an asymptotically stable fixed point,
- (ii) if $Sf(c) > 0$, then $x = c$ is an unstable fixed point.

Proof. (i) Set $g(x) = f^2(x)$, then $g(c) = c$ and we see that if c is asymptotically stable with respect to g , then it is asymptotically stable with respect to f . Now

$$g'(x) = \frac{d}{dx}(f(f(x))) = f'(f(x)) \cdot f'(x),$$

so that $g'(c) = f'(c) \cdot f'(c) = (-1)(-1) = 1$.

The idea is to apply Theorem 1.5.4 (iii) to the function $g(x)$. Now

$$g''(x) = f'(f(x)) \cdot f''(x) + f''(f(x)) \cdot [f'(x)]^2,$$

thus

$$g''(c) = f'(c)f''(c) + f''(c)[f'(c)]^2 = 0, \quad \text{since } f'(c) = -1.$$

Also

$$g'''(x) = f''(f(x)) \cdot f'(x) \cdot f''(x) + f'(f(x)) \cdot f'''(x) + f'''(f(x)) [f'(x)]^3 + f''(f(x)) \cdot 2f'(x)f''(x).$$

Therefore

$$\begin{aligned} g'''(c) &= [f''(c)]^2(-1) - f'''(c) - f'''(c) + 2f''(c)(-1)f''(c) \\ &= -2f'''(c) - 3[f''(c)]^2 \\ &= 2Sf(c) < 0, \end{aligned}$$

and the result follows from Theorem 1.5.4 (iii).

(ii) This now follows from Theorem 1.5.4 (ii)

□

Remark 1.5.8 The above proof shows how the Schwarzian derivative arises from differentiating $g = f \circ f = f^2$. In the case where $f'(c) = -1$, it follows that

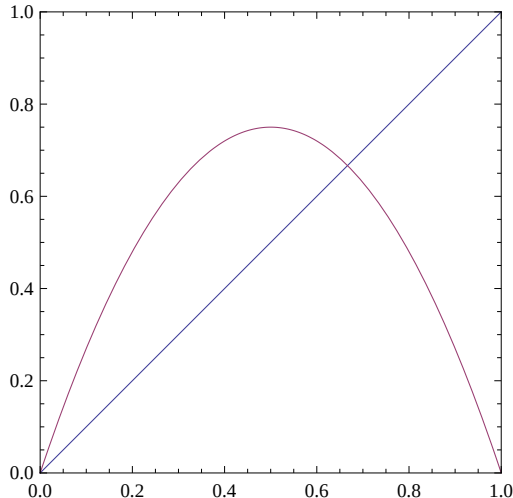
$$g''(c) = 0 \quad \text{and} \quad Sf(c) = \frac{1}{2}g'''(c).$$

Example 1.5.9 For the logistic map $L_\mu(x) = \mu x(1 - x)$ we have

$$L'_\mu(x) = \mu - 2\mu x, \quad L''_\mu(x) = -2\mu \quad \text{and} \quad L'''_\mu(x) = 0.$$

When $\mu = 1$, $x = 0$ is the only fixed point and Theorem 1.5.4 (i) shows that $x = 0$ is semi-stable (attracting on the right). However, we regard this as a stable fixed point for L_μ defined on the interval $[0, 1]$, since points to the left of 0 are not in the domain of L_μ .

When $\mu = 3$, $c = 2/3$ is fixed and $L'_\mu(2/3) = -1$ giving a non-hyperbolic fixed point. However, $Sf(2/3) = 0 - \frac{3}{2}[6]^2 < 0$ (negative Schwarzian derivative), so by Theorem 1.5.7 (i), $x = 2/3$ is asymptotically stable.



Exercises 1.5

1. Show that $f(x) = -2x^3 + 2x^2 + x$ has two non-hyperbolic fixed points and determine their stability.

2. For the family of quadratic maps $Q_c(x) = x^2 + c$, $x \in \mathbb{R}$, use the Theorems of Section 1.5 to determine the stability of the fixed points for all possible values of c . Find any values of c so that Q_c has a non-hyperbolic fixed point, and determine their stability.

3. Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable or unstable:

$$(i) f(x) = \frac{x^3}{2} + \frac{x}{2}, \quad (ii) f(x) = \arctan x, \quad (iii) f(x) = x^3 + x^2 + x,$$

$$(iv) f(x) = x^3 - x^2 + x, \quad (v) f(x) = \begin{cases} 0.8x & ; \text{if } x \leq 1/2 \\ 0.8(1-x) & ; \text{if } x > 1/2 \end{cases}$$

4. Let $f(x) = ax^2 + bx + c$, $a \neq 0$, and p a fixed point of f . Prove the following:

- (i) If $f'(p) = 1$, then p is semistable.
- (ii) If $f'(p) = -1$, then p is asymptotically stable.

5. If $f(x) = \frac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$ is the linear fractional transformation, show that its Schwarzian derivative is $Sf(x) = 0$ for all x in its domain.

6. If $Sf(x)$ is the Schwarzian derivative of $f(x)$, a C^3 function and $F(x) = \frac{f''(x)}{f'(x)}$, show that $Sf(x) = F'(x) - (F(x))^2/2$.

7. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, find the fixed points of f and show that for $x \neq 0$ they are non-hyperbolic. Show that $x = 0$ is not an isolated fixed point (i.e., there are other fixed points arbitrarily close to 0). Is $x = 0$ a stable, attracting or repelling fixed point?

8. ([50]) Let N_f be the Newton function of a four times continuously differentiable function f . If $f(\alpha) = 0$, show that $N_f'''(\alpha) = 2Sf(\alpha)$, where Sf is the Schwarzian derivative of f .
9. (a) Use the Intermediate Value Theorem to show that $f(x) = \cos(x)$ has a fixed point c in the interval $[0, \pi/2]$. We can show experimentally that this fixed point is approximately $c = .739085\dots$, for example by iterating any $x_0 \in \mathbb{R}$.
- (b) Show that the basin of attraction of c is all of $\mathbb{R}$ (Hint: You may assume that $x \in [-1, 1]$ - why? Now use the Mean Value Theorem to show that $|f(x) - c| < \lambda|x - c|$ for some $0 < \lambda < 1$).
- (c) Does $f(x)$ have any eventual fixed points?
- (d) Can $f(x)$ have any points p with $f^2(p) = p$ other than c ?

Chapter 2, Bifurcations and the Logistic Family

2.1 The Basin of Attraction

In this chapter we examine the basin of attraction of the logistic maps, i.e., for a given μ and fixed point $x = c$ we look for the set of those $x \in [0, 1]$ which converge to c under iteration by $L_\mu(x) = \mu x(1 - x)$.

Definition 2.1.1 The *basin of attraction* $B_f(c)$ of a fixed point c of $f(x)$ is the set of all x for which the sequence $x_n = f^n(x)$ converges to c :

$$B_f(c) = \{x \in X : f^n(x) \rightarrow c, \text{ as } n \rightarrow \infty\}.$$

The *immediate basin of attraction* of f is the largest interval containing c , contained in the basin of attraction of c . We first show that this is always an open interval when c is an attracting fixed point.

Proposition 2.1.2 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function having an attracting fixed point p . The immediate basin of attraction of p is an open interval.*

Proof. Since p is an attracting fixed point there is $\epsilon > 0$ such that for all $x \in I_\epsilon = (p - \epsilon, p + \epsilon)$, $f^n(x) \rightarrow p$ as $n \rightarrow \infty$. Denote by J the largest interval containing p for which $f^n(x) \rightarrow p$ for $x \in J$, as $n \rightarrow \infty$.

Suppose that $J = [a, b]$, a closed interval, then there exists $r \in \mathbb{Z}^+$ with $f^r(a) \in I_\epsilon$. Now f^r is also a continuous function, so points close to a will also get mapped into I_ϵ , leading to a contradiction.

More specifically, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f^r(x) - f^r(a)| < \eta$, where $\eta = \min\{|f^r(a) - (p - \epsilon)|, |(p + \epsilon) - f^r(a)|\}$. Thus there are points $x < a$ close to a for which $f^r(x) \in I_\epsilon$, so $f^{rn}(x) \rightarrow p$ as $n \rightarrow \infty$, a contradiction. We conclude that $a \notin J$ (and similarly for b), so J is an open interval. $\square$

Example 2.1.3 1. If $f(x) = x^2$, the fixed points are $c = 0$ and $c = 1$, both hyperbolic, the first being attracting and the second repelling. Clearly $B_f(0) = (-1, 1)$ and $B_f(1) = \{-1, 1\}$. We sometimes regard $c = \infty$ as an attractive fixed point of f , so that $B_f(\infty) = [-\infty, -1) \cup (1, \infty]$.

2. If $f : I \rightarrow I$ is a continuous function on a closed interval $I = [a, b]$ having an attracting fixed point $p \in (a, b)$, then we cannot exclude the possibility that the basin of attraction includes either a or b , or both. On the other hand, if $x = a$ is the attracting fixed point, the basin of attraction may be a set of the form $[a, c)$ for some

$c \in (a, b]$. Such a set can be regarded as being open as a subset of $[a, b]$ (see Chapter 4).

2.2 The Logistic Family

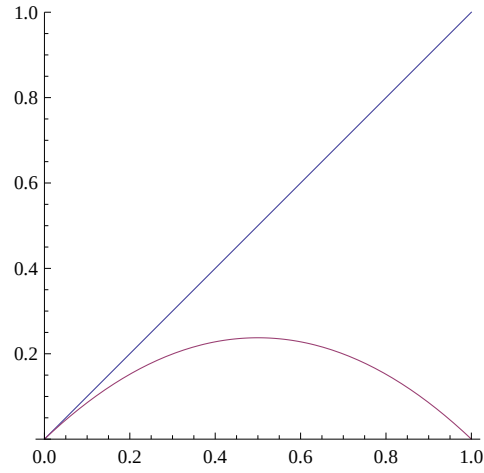
The logistic maps $L_\mu(x) = \mu x(1 - x)$ are functions of two real variables μ and x . We usually restrict x to the interval $[0, 1]$ and consider $\mu \in (0, 4]$. μ is a parameter which we allow to vary, but then we study the function L_μ for specific fixed values of μ . As the parameter μ is varied, we see a corresponding change in the nature of the function L_μ . This is what is called *bifurcation*. For example, for $0 < \mu \leq 1$, L_μ has exactly one fixed point in $[0, 1]$, $c = 0$, which is attracting. As μ increases beyond 1, a new fixed point $c = 1 - 1/\mu$, is created in $[0, 1]$, so now L_μ has two fixed points. $c = 0$ is now repelling and $c = 1 - 1/\mu$ is attracting (for $1 < \mu \leq 3$). At $\mu = 3$ the nature of these fixed points changes again as we shall see. In this section we determine the basin of attraction of these fixed points as μ increases from 0 to 3. We see that the “dynamics” (long term behavior) of L_μ is quite uncomplicated for this range of values of μ .

The function $L_\mu(x) = \mu x(1 - x)$, $0 \leq x \leq 1$ has a maximum value of $\mu/4$ when $x = 1/2$. Consequently, for $0 < \mu \leq 4$, L_μ maps the unit interval $[0, 1]$ into itself. We shall consider later what happens when $\mu > 4$. We start by showing that the basin of attraction of L_μ for $0 < \mu \leq 1$ is all of the domain of L_μ , namely $[0, 1]$. We say that 0 is a *global attractor* in this case.

Theorem 2.2.1 *Let $L_\mu(x) = \mu x(1 - x)$, $0 \leq x \leq 1$ be the logistic map. For $0 < \mu \leq 1$, $B_{L_\mu}(0) = [0, 1]$ and for $1 < \mu \leq 3$, $B_{L_\mu}(1 - 1/\mu) = (0, 1)$.*

We split the proof into a number of different cases:

Case 2.2.2 $0 < \mu \leq 1$.



For $0 < \mu < 1$, the only fixed point is 0.

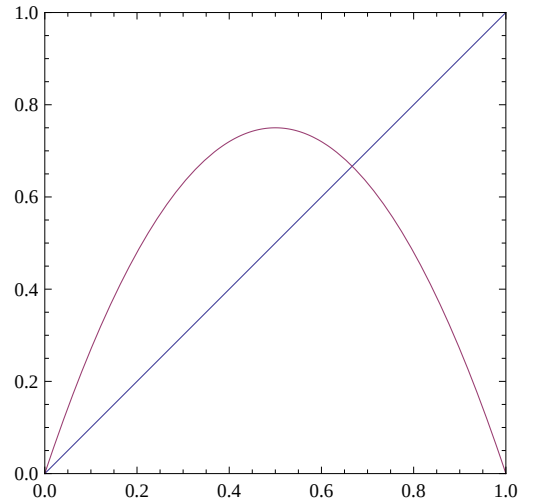
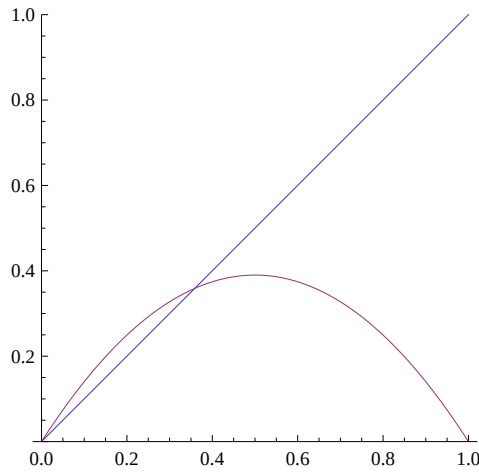
We have seen that for $\mu \in (0, 1)$, L_μ has only the one fixed point $x = 0$ in $[0, 1]$ (the other fixed point is $1 - 1/\mu \leq 0$). For $\mu < 1$ this fixed point is asymptotically stable, but in any case

$$0 < \mu \leq 1, \quad 0 < 1 - x < 1 \Rightarrow 0 < \mu(1 - x) < 1$$

$$\Rightarrow 0 < L_\mu(x) = \mu x(1 - x) < x, \quad x \in (0, 1],$$

and in a similar way, $L_\mu^2(x) < L_\mu(x)$ etc., so the sequence $L_\mu^n(x)$ is decreasing, bounded below by 0, and hence must converge to the only fixed point, namely 0. It follows that the basin of attraction is $B_{L_\mu}(0) = [0, 1]$ ($L_\mu(1) = 0$).

Case 2.2.3 $1 < \mu \leq 3$.



$$1 - 1/\mu < 1/2 \text{ for } 1 < \mu < 2.$$

$$1 - 1/\mu \geq 1/2 \text{ for } 2 \leq \mu \leq 3.$$

We have seen that for $\mu > 1$ the fixed point 0 is repelling, but a new fixed point $c = 1 - 1/\mu$ has been “born” which is attracting (for $1 < \mu \leq 3$), so by Proposition 2.1.2 there is a largest open interval $I = (a, b)$ containing the fixed point for which $x \in I$ implies $L_\mu^n(x) \rightarrow c$ as $n \rightarrow \infty$. If the basin of attraction of c is $B_\mu(c)$, then $0, 1 \notin B_\mu(c)$ because $L_\mu(0) = 0$ and $L_\mu(1) = 0$, so $B_\mu(c) \neq [0, 1]$. Furthermore, clearly $a, b \notin B_\mu(c)$.

From the Intermediate Value Theorem, $L_\mu(a, b)$ is an interval which must be contained in (a, b) , for if $x \in (a, b)$, $L_\mu^n(L_\mu(x)) \rightarrow c$ as $n \rightarrow \infty$. Let x_n be a sequence in (a, b) with $\lim_{n \rightarrow \infty} x_n = a$, then by the continuity of L_μ , $\lim_{n \rightarrow \infty} L_\mu(x_n) = L_\mu(a)$. Since $x_n \in (a, b)$ for every $n \in \mathbb{Z}^+$, we have $L_\mu(x_n) \in (a, b)$ for all $n \in \mathbb{Z}^+$. Since $L_\mu(a) \notin (a, b)$, the only way this is possible is if $L_\mu(a) = a$ or $L_\mu(a) = b$, and similarly for b . This is only possible if a and b are fixed points, are eventual fixed points or $L_\mu(a) = b, L_\mu(b) = a$ (we call $\{a, b\}$ a 2-cycle - see Section 2.3). We show that the latter case cannot happen, and this leads to the conclusion that $a = 0$ and $b = 1$, since there are no other fixed or eventual fixed points in $[0, 1]$ that can satisfy these condition. Consequently, we must have $B_\mu(1 - 1/\mu) = (0, 1)$.

Now we can check that

$$L_\mu^2(x) - x = \mu^2 x(1 - (\mu + 1)x + 2\mu x^2 - \mu x^3) - x = x(\mu^2 x^2 - \mu(\mu + 1)x + \mu + 1)(\mu x - \mu + 1),$$

and a and b must satisfy this equation. We can disregard the linear factors as they give the two fixed points. The discriminant of the quadratic factor is $\mu^2(\mu + 1)^2 - 4\mu^2(\mu + 1) = \mu^2(\mu + 1)(\mu - 3) < 0$ for $1 < \mu < 3$, so there is no 2-cycle when $1 < \mu < 3$. When $\mu = 3$, the discriminant is zero, the fixed point is $c = 2/3$ and the quadratic factor gives rise to no additional roots, so again there is no 2-cycle. $\square$

Remark 2.2.4 Note that when $\mu = 2$, $x = 1/2$ is a super attracting fixed point (since $L'_2(1/2) = 0$). As we saw above the basin of attraction will be $(0, 1)$.

Exercises 2.2

1. If $L_\mu(x) = \mu x(1 - x)$ is the logistic map, show that $x = 1/2$ is the only turning point of $L^2(x)$ for $0 < \mu \leq 2$, but when $\mu > 2$, two new turning points are created.

Use this to show that for $2 < \mu < 3$, the interval $[1/\mu, 1 - 1/\mu]$ is mapped by L_μ^2 onto the interval $[1/2, 1 - 1/\mu]$.

2.3 Periodic Points

Points with finite orbits are of importance in the study of dynamical systems and their long-term behavior:

Definition 2.3.1 Let $f : X \rightarrow X$ be a function with $c \in X$.

(i) c is a *periodic point* of $f(x)$ with period $r \in \mathbb{Z}^+$ if $f^r(c) = c$ and $f^k(c) \neq c$ for $0 < k < r$ (in particular, c is a fixed point of f^r). We call r the *period* of c and the set $O(c) = \{c, f(c), f^2(c), \dots, f^{r-1}(c)\}$ is an *r-cycle*. We write

$$\text{Per}_r(f) = \{x \in X : f^r(x) = x\},$$

so that $\text{Fix}(f) \subseteq \text{Per}_n(f)$, $n = 1, 2, \dots$, since the points in $\text{Per}_n(f)$ may not be of period n , but of some lesser period.

(ii) c is *eventually periodic* for f if there exists $m \in \mathbb{Z}^+$ such that $f^m(c)$ is a periodic point of f (we assume that c is not a periodic point).

(iii) c is *stable* (respectively *asymptotically stable*, *unstable* etc.) if it is a stable fixed point of f^r .

The following criteria for stability now immediately follows from Theorem 1.3.3:

Theorem 2.3.2 Suppose that c is a point of period r for f and that $f'(x)$ is continuous at $x = c$. If $c_i = f^i(c)$, $i = 0, 1, \dots, r - 1$ then:

(i) c is asymptotically stable if

$$|f'(c_0) \cdot f'(c_1) \cdot f'(c_2) \cdots f'(c_{r-1})| < 1.$$

(ii) c is unstable if

$$|f'(c_0) \cdot f'(c_1) \cdot f'(c_2) \cdots f'(c_{r-1})| > 1.$$

Proof. Let us look at the case where $r = 3$ as this is typical;

$$O(c) = \{c, f(c), f^2(c)\} = \{c_0, c_1, c_2\}.$$

Then

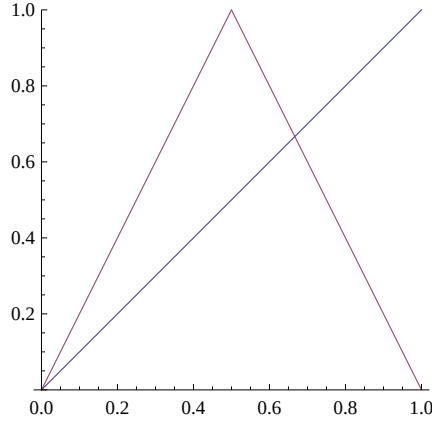
$$\frac{d}{dx}(f^3(x)) = \frac{d}{dx}(f(f^2(x))) = f'(f^2(x))(f^2(x))' = f'(f^2(x))f'(f(x))f'(x)$$

$$= f'(c_2)f'(c_1)f'(c_0), \quad \text{when } x = c.$$

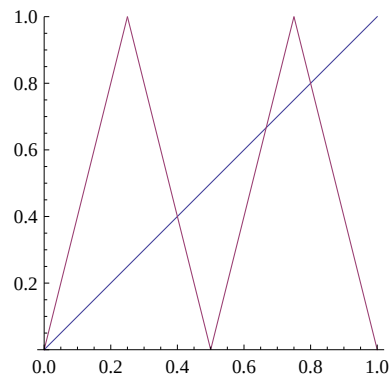
The result now follows from 1.3.3. □

Example 2.3.3 If we look at the graph of the tent map $T(x)$, we see it has two fixed points, $c = 0$ and $c = 2/3$. If we graph T^3 it has eight fixed points: These arise from two 3-cycles: $\{2/7, 4/7, 6/7\}$ and $\{2/9, 4/9, 8/9\}$, together with the two fixed points, so that

$$\text{Per}_3(T) = \{0, 2/3, 2/7, 4/7, 6/7, 2/9, 4/9, 8/9\}.$$



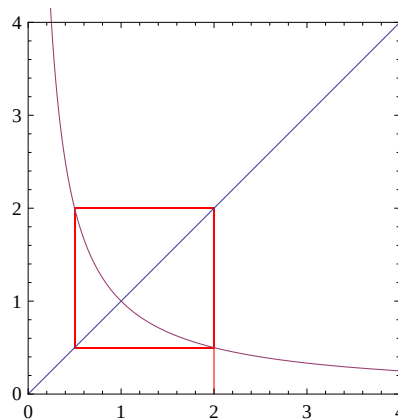
The graph of T^2 shows that T^2 has four fixed points coming from a 2-cycle $\{2/5, 4/5\}$ and the two fixed points. Since $|T'(x)| = 2$ and $|(T^2)'(x)| = |T'(x)||T'(Tx)| = 4$, etc. (except at points of non-differentiability), all these periodic points will be unstable.



Example 2.3.4 Let $f(x) = 1/x$, $x \neq 0, x \neq \pm 1$. Note that $f^2(x) = x$ and $f(x) \neq x$ for all such x , giving rise to the 2-cycle $\{x, 1/x\}$. In this case

$$|f'(x)f'(1/x)| = |-1/x^2(-x^2)| = 1,$$

so the theorem is inconclusive. However we see that the periodic points are stable but neither attracting nor repelling.



Example 2.3.5 Consider the quadratic function $f(x) = x^2 - 2$. We have seen that to find the fixed points we solve $f(x) = x$, or $x^2 - 2 = x$, $x^2 - x - 2 = (x - 2)(x + 1) = 0$, so $x = 2$ or $x = -1$.

To find the period 2-points we solve $f^2(x) = x$ or $f^2(x) - x = 0$. This is simplified when we realize that the fixed points must be solutions of this equation, so that $(x - 2)(x + 1)$ is a factor. We can then check that

$$f^2(x) - x = x^4 - 4x^2 - x + 2 = (x - 2)(x + 1)(x^2 + x - 1).$$

Solving the quadratic gives

$$x = \frac{-1 \pm \sqrt{5}}{2},$$

so that $\{\frac{-1 + \sqrt{5}}{2}, \frac{-1 - \sqrt{5}}{2}\}$ is a 2-cycle. In general, finding periodic points of quadratics can be complicated. If $f(x)$ is a quadratic, $f^n(x) - x$ is a polynomial of degree 2^n .

To check the stability, we calculate

$$|f'((-1 - \sqrt{5})/2)f'((-1 + \sqrt{5})/2)| = |(-1 - \sqrt{5})(-1 + \sqrt{5})| = |1 - 5| = 4 > 1,$$

giving an unstable 2-cycle.

Remark 2.3.6 1. As before, periodic points can be stable but not attracting (as above with $f(x) = 1/x$ at $x \neq 1$). They can also be attracting but not stable as in Example 1.4.7).

2. Functions such as $f(x) = \sin x$ can have no period 2 points or points of a higher period since this would contradict the basin of attraction of $x = 0$ being all of $\mathbb{R}$. Similarly, the logistic map L_μ , $0 < \mu \leq 3$ cannot have period n -points for $n > 1$.

Exercises 2.3

1. For each of the following functions, $c = 0$ lies on a periodic cycle. Classify this cycle as attracting, repelling or neutral (non-hyperbolic). Say if it is super-attracting:

$$(i) \ f(x) = \frac{\pi}{2} \cos x, \quad (ii) \ g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

2. (a) Show that $C_\mu(x) = \mu \cos(x)$ has a super-attracting 3-cycle $\{0, \lambda, \pi/2\}$ where $\mu = \lambda$ and λ satisfies the equation $\lambda \cos(\lambda) = \pi/2$.

(b) Give similar conditions for $S_\mu(x) = \mu \sin(x)$ to have (i) a super-attracting 2-cycle, (ii) a super-attracting 3-cycle.

(c) Explain why for families of maps, say F_μ , one member of a super-attracting n -cycle is a super-attracting fixed point (for a different value of μ).

3. Find the fixed points and the period two points of following maps (if any) and determine the stability of the 2-cycle:

$$(i) f(x) = \frac{x^2}{2} - x + \frac{1}{2}, \quad (ii) f(x) = a - \frac{b}{x} \quad (b \neq 0), \quad (iii) f(x) = \frac{1-x}{3x+1}, \quad (iv) f(x) = |x-1|.$$

4. Let $Q_c(x) = x^2 + c$. Show that for $c < -3/4$, Q_c has a 2-cycle and find it explicitly. For what values of c is the 2-cycle attracting?

5. Let $f(x) = ax^2 + bx + c$, $a \neq 0$. Let $\{x_0, x_1\}$ be a 2-cycle for $f(x)$,

(a) If $f'(x_0)f'(x_1) = -1$, prove that the 2-cycle is asymptotically stable.

(b) If $f'(x_0)f'(x_1) = 1$, prove that the 2-cycle is asymptotically stable.

6. Let $f(x) = ax^3 - bx + 1$, $a \neq 0$. If $\{0, 1\}$ is a 2-cycle for $f(x)$, give conditions on a and b under which it is asymptotically stable.

7. Let $f(x)$ be a polynomial with $f(c) = c$.

(i) If $f'(c) = 1$, show that $(x - c)^2$ is a factor of $g(x) = f(x) - x$.

(ii) If $f'(c) = 1$, or $f'(c) = -1$, show that $(x - c)^2$ is a factor of $h(x) = f^2(x) - x$ (i.e., if $f(x)$ has a non-hyperbolic fixed point c , then c is a repeated root of $f^2(x) - x$). Hint: Recall that a polynomial $p(x)$ has $(x - c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.

(iii) Show in the case that $f'(c) = -1$ we actually have that $(x - c)^3$ is a factor of $h(x) = f^2(x) - x$.

(iv) Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$, of the logistic map $L_3(x) = 3x(1 - x)$.

(v) Check that (i), (ii) and (iii) hold for the (non-hyperbolic) fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function which is differentiable everywhere. If $\{x_0, x_1, \dots, x_{n-1}\}$ is an n -cycle for f , show that the derivative of f^n is the same at each x_i , $i = 0, 1, \dots, n-1$.

9. Show that if x_0 is a periodic point of f with period n and $f^m(x_0) = x_0$ $m \in \mathbb{Z}^+$, then $m = kn$ for some $k \in \mathbb{Z}^+$. (Hint: Write $m = qn + r$ for some r , $0 \leq r < n$ and show that $r = 0$).

10. Show that if $f^p(x) = x$ and $f^q(x) = x$, and n is the highest common factor of p and q , then $f^n(x) = x$. (Hint: Use the previous question and the fact that every common factor of p and q is a factor of n).

11. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an *odd function* ($f(-x) = -f(x)$ for all $x \in \mathbb{R}$). Show that $x = 0$ is a fixed point and that the intersection of the graph of f with the line $y = -x$ gives rise to points c of period 2 (when $c \neq 0$).

(b) Use part (a) to find the 2-cycles of $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - 3x/2$ and determine their stability.

(c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an *even function* ($f(-x) = f(x)$ for all $x \in \mathbb{R}$). Show that the intersection of the graph of f with the line $y = -x$ gives rise to eventually fixed points c (when $c \neq 0$). What are the eventually fixed points of $f(x) = \cos(x)$?

12. (a) Let $f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. In Exercises 1.7. # 7, we found the fixed points of f and showed that the non-zero fixed points are non-hyperbolic. Find the eventual fixed points of f (note that $f(x)$ is an even function).

(b) (Open ended question) Discuss the stability of the fixed points of $f(x)$, their basins of attractions and the existence of period 2-points. Note that $f(x) \leq x$ for all $x > 0$.

(c) Now set $g(x) = \begin{cases} x \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Find the fixed points of g , and show that in this case there are period 2-points (Hint: note that $g(x)$ is an odd function).

2.4 Periodic Points of the Logistic Map

We try to find the 2-cycles of the logistic map $L_\mu(x) = \mu x(1 - x)$, $0 \leq x \leq 1$. To do this we solve the equation

$$L_\mu^2(x) = x,$$

or

$$\mu x(1 - x)[1 - \mu x(1 - x)] - x = 0,$$

or

$$-\mu^3 x^4 + 2\mu^3 x^3 - (\mu^3 + \mu^2)x^2 + \mu^2 x - x = 0,$$

Clearly x is a factor (since $c = 0$ is a fixed point of $L_\mu(x)$) and similarly $x - (1 - 1/\mu)$ must be a factor so we get

$$L_\mu^2(x) - x = -x(\mu x - \mu + 1)(\mu^2 x^2 - \mu(\mu + 1)x + \mu + 1),$$

giving a quadratic equation which has no roots if $\mu < 3$:

$$\mu^2 x^2 - \mu(\mu + 1)x + \mu + 1 = 0.$$

Solving using the quadratic formula gives

$$\begin{aligned} c &= \frac{\mu(\mu + 1) \pm \sqrt{\mu^2(\mu + 1)^2 - 4\mu^2(\mu + 1)}}{2\mu^2} \\ &= \frac{(1 + \mu) \pm \sqrt{(\mu - 3)(\mu + 1)}}{2\mu}. \end{aligned}$$

This is real only for $\mu \geq 3$ (called the “birth of period two”). Let us call these two roots c_1 and c_2 (dependent on μ).

This 2-cycle is asymptotically stable if

$$|(L_\mu^2)'(c_1)| = |L_\mu'(c_1)L_\mu'(c_2)| < 1,$$

or

$$\begin{aligned} -1 &< \mu^2(1 - 2c_1)(1 - 2c_2) < 1, \\ -1 &< (-1 - \sqrt{(\mu^2 - 2\mu - 3)})(-1 + \sqrt{(\mu^2 - 2\mu - 3)}) < 1, \\ -1 &< 1 - (\mu^2 - 2\mu - 3) < 1, \end{aligned}$$

and this gives rise to the two inequalities

$$\mu^2 - 2\mu - 3 > 0 \quad \text{and} \quad \mu^2 - 2\mu - 5 < 0,$$

and solving gives

$$3 < \mu < 1 + \sqrt{6}.$$

This is the condition for asymptotic stability of the 2-cycle $\{c_1, c_2\}$.

For $\mu = 1 + \sqrt{6}$, it can be seen that

$$L'_\mu(c_1)L'_\mu(c_2) = -1, \quad \text{and} \quad SL_\mu^2(c_1) < 0,$$

so Theorem 1.5.7 (i) shows that the 2-cycle is asymptotically stable. Also, the 2-cycle is unstable for $\mu > 1 + \sqrt{6}$. In summary:

Theorem 2.4.1 *For $3 < \mu \leq 1 + \sqrt{6}$, the logistic map $L_\mu(x) = \mu x(1 - x)$ has an asymptotically stable 2-cycle. The 2-cycle is unstable for $\mu > 1 + \sqrt{6}$.*

The above shows that we have a bifurcation when $\mu = 3$ where a 2-cycle is created and which was not previously present. There is another bifurcation at $\mu = 1 + \sqrt{6}$. This means that for $3 < \mu \leq 1 + \sqrt{6}$, when we use graphical iteration of points close to c_1 and c_2 , they will approach the period 2 orbit, and not the fixed point (which is now unstable). In fact it can be shown that for this range of values of μ , the basin of attraction of the 2-cycle consists of all of $(0, 1)$, (except for the fixed point $1 - 1/\mu$ and eventual fixed points such as $1/\mu$). When μ exceeds $1 + \sqrt{6} = 3.449499\dots$, the period 2-points become unstable and this no longer happens, but we shall see that something different happens. We have another bifurcation when $\mu = 1 + \sqrt{6}$, with the birth of an attracting period 4-cycle.

2.5 The Period Doubling Route to Chaos

We summarize what we have determined so far and talk about what happens as μ increases from zero to around 3.57.

For $\mu < b_1 := 1$, $c = 0$ is the only fixed point and it is attracting for these values of μ . There is a bifurcation at $b_1 = 1$, where a non-zero fixed point $c = 1 - 1/\mu$ is created. This fixed point is attracting for $\mu \leq 3$ (and $c = 0$ is no longer attracting) and super attracting when $\mu = s_1 = 2$. The second bifurcation occurs when $\mu = b_2 = 3$. The fixed point $c = 1 - 1/\mu$ becomes unstable, and an attracting 2-cycle is created for $3 < \mu < 1 + \sqrt{6} = b_3$.

2.5.1 A Super-Attracting Period 2-Cycle

We saw that when $\mu = 2$, $c = 1/2$ is a super-attracting fixed point for L_μ . We now look for a super-attracting period 2-cycle for L_μ when $3 < \mu < 1 + \sqrt{6}$, as it illustrates an important general method that can be used for finding where period three is born:

Suppose that $\{x_1, x_2\}$ is a 2-cycle for the logistic map L_μ , which is super-attracting, then

$$x_1 = \mu x_2(1 - x_2), \quad \text{and} \quad x_2 = \mu x_1(1 - x_1),$$

so multiplying these equations together gives the equation

$$\mu^2(1 - x_1)(1 - x_2) = 1.$$

In addition we must have

$$(L_\mu^2)'(x_1) = L_\mu'(x_1)L_\mu'(x_2) = 0,$$

so that

$$\mu^2(1 - 2x_1)(1 - 2x_2) = 0.$$

Thus either $x_1 = 1/2$, or $x_2 = 1/2$, so suppose the former holds, then $x_2 = \mu/4$. Substituting into the first equation gives

$$\mu^2(1 - \mu/4)(1 - 1/2) = 1,$$

or

$$\mu^3 - 4\mu^2 + 8 = 0.$$

$\mu - 2$ must be a factor of this cubic, so we have

$$(\mu - 2)(\mu^2 - 2\mu - 4) = 0,$$

and this gives $\mu = 1 + \sqrt{5}$. We have shown:

Proposition 2.5.2 *When $\mu = s_2 = 1 + \sqrt{5}$, $\{1/2, \frac{1 + \sqrt{5}}{4}\}$ is a super-attracting 2-cycle for L_μ .*

When μ exceeds $b_3 = 1 + \sqrt{6}$, the 2-cycle ceases to be attracting and becomes repelling. In addition, a 4-cycle is created which is attracting until μ exceeds a value b_4 when it becomes repelling and an attracting 8-cycle is created. This type of period doubling continues so that when μ exceeds b_n , an attracting 2^{n-1} -cycle is created until μ reaches b_{n+1} . These cycles become super attracting at some s_n ($b_n < s_n < b_{n+1}$). This behavior continues with 2^n -cycles for all $n \in \mathbb{Z}^+$ being created, until μ reaches approximately 3.57. In other words, for $b_n < \mu < b_{n+1}$, L_μ has a stable 2^n -cycle. It can be shown that $b_\infty = \lim_{n \rightarrow \infty} b_n = 3.570$ approximately.

Although $b_{n+1} - b_n \rightarrow 0$ as $n \rightarrow \infty$, it can be shown that

$$\lim_{n \rightarrow \infty} \frac{b_n - b_{n-1}}{b_{n-1} - b_n} = \delta = 4.6692016 \dots,$$

(called Feigenbaum's number). Feigenbaum showed that you get the same constant δ in this way for any family of unimodal maps ($f_\mu(x)$ is *unimodal* if $f_\mu(0) = 0$, $f_\mu(1) = 0$, f_μ is continuous on $[0, 1]$ with a single critical point between 0 and 1).

2.6 The Bifurcation Diagram.

The behavior described above can be illustrated graphically using a *bifurcation diagram*. To create a bifurcation diagram we plot μ , $0 \leq \mu \leq 4$ along the x -axis, and values of $L_\mu^n(x)$ along the y -axis. The idea is to calculate (say) for each value of μ the first 500 iterates of some (arbitrarily chosen) point x_0 . We ignore the first 450 iterates and plot the next 50. So for example, if $1 < \mu < 3$, because the fixed point is attracting, the iterates will approach the fixed point $1 - 1/\mu$, so for n large, what we see plotted will be (very close to) the value $1 - 1/\mu$. For $3 < \mu < 1 + \sqrt{6}$ the fixed point has become repelling, so this no longer shows up, but the 2-cycle has become attracting, so we see plotted the 2 points of the 2-cycle. This continues with the 4-cycle, 8-cycle etc. This is called the *period doubling route to chaos*.

We can create a bifurcation diagram for L_μ using Mathematica in the following way:

First we define the logistic map for $\mu = 1$, and then a function of two variables:

```
f[x_]:= x(1-x)

h[x_,a_]:= a*f[x]
```

For a given $a \in [0, 4]$ we pick $x \in (0, 1)$ randomly and iterate $h(x, a)$ 100 times. Since this gives rise to a single number, we can define a function as the output:

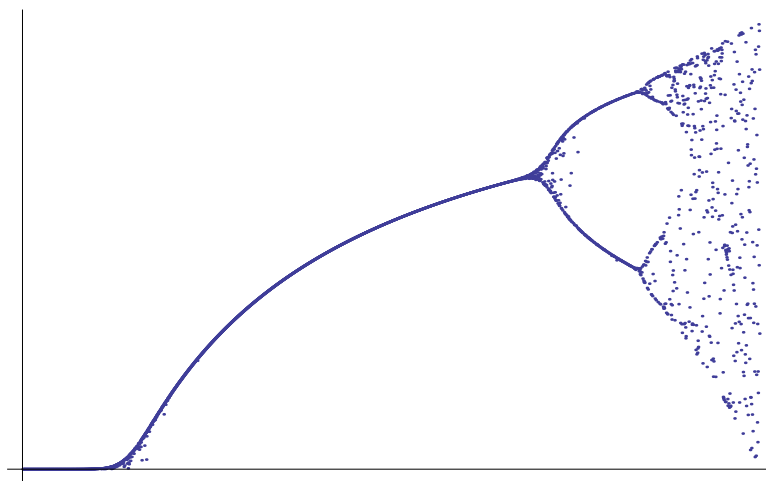
```
g[a_]:= (
  k[x_]:=h[x,a];
  A=NestList[k, Random[Real, {0,1}], 100];
  Return[A[[100]])
)
```

Now we generate a list using the `Table` command for $a \in [1, 4]$:

```
B=Table[g[j/1000], {j, 1000, 4000}];
```

and now we plot this list:

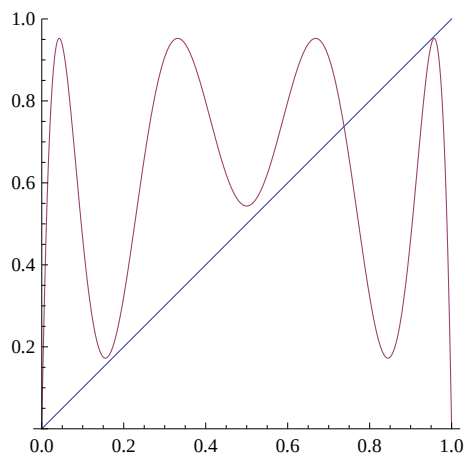
```
ListPlot[B, PlotStyle->PointSize[0.001], Ticks->False]
```



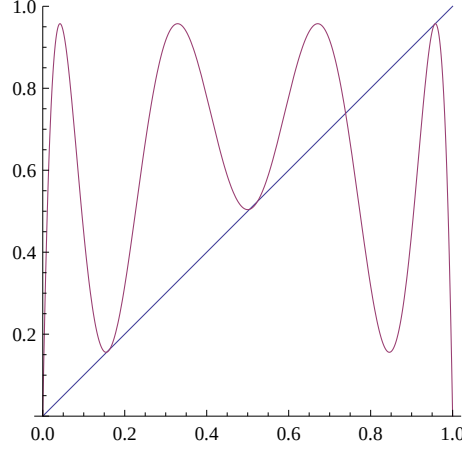
The bifurcation diagram for the logistic map.

2.6.1 Where Does Period Three Occur for the Logistic Map

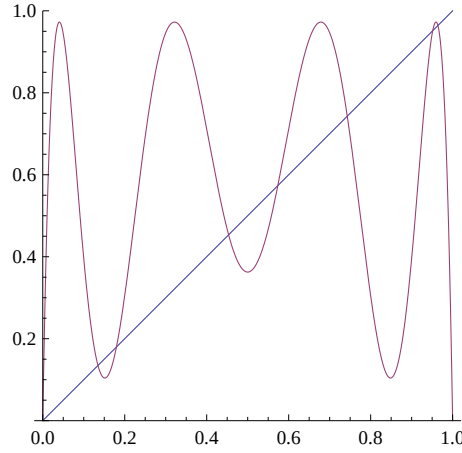
If we look at the graph of L_μ^3 for values of μ close to 3.8, we see that is where a 3-cycle is created:



The Graph of $L_\mu^3(x)$ for $\mu < 1 + \sqrt{8}$.



The Graph of $L_\mu^3(x)$ for $\mu = 1 + \sqrt{8}$.



The Graph of $L_\mu^3(x)$ for $\mu > 1 + \sqrt{8}$.

We aim to show that this happens when $\mu = 1 + \sqrt{8}$. For values of μ slightly smaller than this we only see the two fixed points. For $\mu = 1 + \sqrt{8}$ we see that a 3-cycle is born, and for larger values of μ , we see two 3-cycles. For a small range of values of μ , the 3-cycle is attracting.

To show that the bifurcation occurs when $\mu = 1 + \sqrt{8}$ we follow the argument of Feng [24] (see also [7], [27] and [56]):

Period 3-points of L_μ occur where $L_\mu^3(x) = x$, so we look at where

$$L_\mu^3(x) - x = 0.$$

To disregard the fixed points of L_μ we set

$$g_\mu(x) = \frac{L_\mu^3(x) - x}{L_\mu(x) - x}.$$

Using Mathematica (for example) we can check that

$$\begin{aligned} g_\mu(x) = & \mu^6 x^6 - (\mu^5 + 3\mu^6)x^5 + (\mu^4 + 4\mu^5 + 3\mu^6)x^4 - (\mu^3 + 3\mu^4 + 5\mu^5 + \mu^6)x^3 \\ & + (\mu^2 + 3\mu^3 + 3\mu^4 + 2\mu^5)x^2 - (\mu + 2\mu^2 + 2\mu^3 + \mu^4)x + 1 + \mu + \mu^2. \end{aligned}$$

Set $\lambda = 7 + 2\mu - \mu^2$ and let

$$h_\mu(z) = g_\mu(-z/\mu),$$

then

$$\begin{aligned} h_\mu(z) = & z^6 + (3\mu + 1)z^5 + (3\mu^2 + 4\mu + 1)z^4 + (\mu^3 + 5\mu^2 + 3\mu + 1)z^3 \\ & + (2\mu^3 + 3\mu^2 + 3\mu + 1)z^2 + (\mu^3 + 2\mu^2 + 2\mu + 1)z + \mu^2 + \mu + 1. \end{aligned}$$

Then if

$$k_\mu(z) = \left\{ z^3 + z^2 \frac{(3\mu + 1)}{2} + z(2\mu + 3 - \frac{\lambda}{2}) + \frac{(\mu + 5)}{2} - \frac{\lambda}{2} \right\}^2 + \frac{\lambda}{4}(z + 1)^2(z + \mu)^2,$$

then we can check that $k_\mu(z) = h_\mu(z)$ for all z (e.g., using Mathematica).

Note that

$$\lambda > 0 \quad \text{for} \quad \mu < 1 + \sqrt{8}, \quad \lambda = 0 \quad \text{for} \quad \mu = 1 + \sqrt{8}, \quad \text{and} \quad \lambda < 0 \quad \text{for} \quad \mu > 1 + \sqrt{8}.$$

This means that for $\mu < 1 + \sqrt{8}$, $h_\mu(z) > 0$ for all z (we say $h_\mu(z)$ is *positive definite*), so cannot have any roots, i.e., $g_\mu(x) = 0$ has no solution, so L_μ cannot have any 3-cycles. We summarize this as follows:

Theorem 2.6.2 [24] (i) *If $0 < \mu < 1 + \sqrt{8}$, then $h_\mu(z)$ is positive definite and the equation $h_\mu(z) = 0$ does not have any real roots. Consequently, the logistic map $L_\mu(x)$ does not have a 3-cycle.*

(ii) *If $\mu = 1 + \sqrt{8}$, then $h_\mu(z) = 0$ has three distinct roots, each of multiplicity two. These three roots constitute a 3-cycle for $L_\mu(x)$.*

(iii) *If $\mu > 1 + \sqrt{8}$ (with $\mu - (1 + \sqrt{8})$ sufficiently small), the equation $h_\mu(z) = 0$ has six simple roots which give rise to two 3-cycles for $L_\mu(x)$.*

Proof. (i) If $\mu < 1 + \sqrt{8}$, then since $h_\mu(z)$ is positive definite the result follows.

(ii) If $\mu = 1 + \sqrt{8}$, then $\lambda = 0$ and the equation becomes

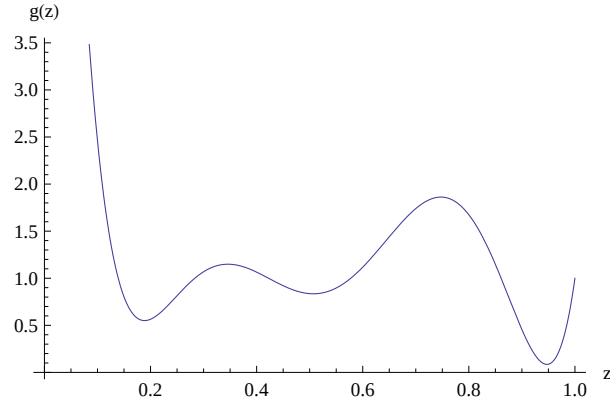
$$h_\mu(z) = \left(z^3 + (2 + 3\sqrt{2})z^2 + (5 + 4\sqrt{2})z + 3 + \sqrt{2} \right)^2.$$

The resulting cubic can be solved using the cubic formula to give three real solutions, z_1, z_2, z_3 and these can be used to give the three solutions to $g_\mu(x) = 0$, corresponding to the 3-cycle of $L_\mu(x)$:

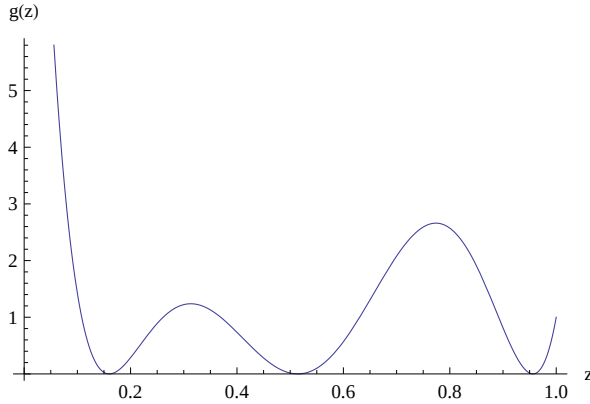
$$z_k = \frac{2\sqrt{7}}{3} \cos \left(\frac{1}{3} \arccos \left(-\frac{1}{2\sqrt{7}} + \frac{2k\pi}{3} \right) \right) - \frac{2+3\sqrt{2}}{3}, \quad k = 0, 1, 2,$$

(see the graph of $g_\mu(x)$ below).

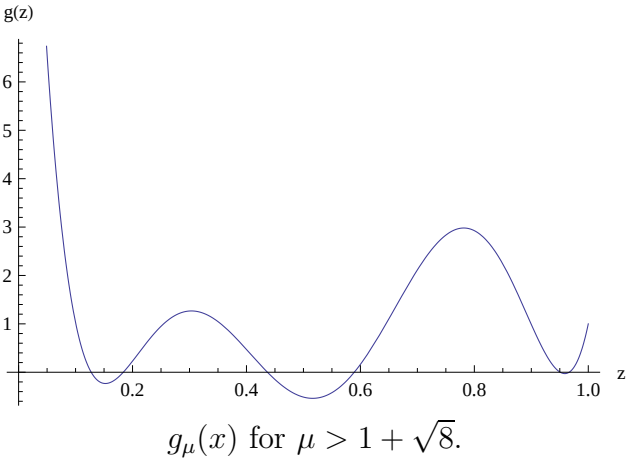
(iii) For $\lambda < 0$ we can factor $h_\mu(z) = h_1(z)h_2(z)$ using the difference of two squares, and then use the Intermediate Value Theorem on each of $h_1(z)$ and $h_2(z)$, to see that they each have three different roots corresponding to two 3-cycles, which can then be shown to be distinct (see [24] for details). $\square$



$g_\mu(x)$ for $\mu < 1 + \sqrt{8}$.



$g_\mu(x)$ for $\mu = 1 + \sqrt{8}$.



Example 2.6.3: A Super-Attracting 3-Cycle for the Logistic Map

Recall that super-attracting periodic points occur where the derivative is zero, so for a super-attracting 3-cycle $\{c_1, c_2, c_3\}$, we require

$$(L_\mu^3)'(c_1) = L_\mu'(c_1)L_\mu'(c_2)L_\mu'(c_3) = 0,$$

i.e.,

$$(1 - 2c_1)(1 - 2c_2)(1 - 2c_3) = 0,$$

so we may assume $c_1 = 1/2$. This means that $x = 1/2$ is a solution of the equation $L_\mu^3(x) = x$, or $L_\mu^3(1/2) = 1/2$. But if μ satisfies the equation $L_\mu(1/2) = 1/2$, then it will also satisfy the equation involving the third iterate. Consequently we solve for μ : $g_\mu(1/2) = 1/2$, where $g_\mu(x) = (L_\mu^3(x) - x)/(L_\mu(x) - x)$ as defined in the last section (this eliminates the root $\mu = 2$ which gave rise to the superstable fixed point at $x = 1/2$). We obtain

$$\frac{1}{64}(64 + 32\mu + 16\mu^2 - 24\mu^3 - 4\mu^4 + 6\mu^5 - \mu^6) = 0.$$

Set

$$p(a) = a^6 - 6a^5 + 4a^4 + 24a^3 - 16a^2 - 32a - 64,$$

then Mathematica indicates that there is a single real root μ_0 larger than $1 + \sqrt{8}$ with exact value

$$\mu_0 = \frac{1}{6} \left\{ 6 + 2\sqrt{3 \left(11 + 2 \cdot 2^{2/3}(25 - 3\sqrt{69})^{1/3} + 2 \cdot 2^{2/3}(25 + 3\sqrt{69})^{1/3} \right)} \right\}.$$

Mathematica is able to solve this equation exactly because it can be reduced to a cubic (and then the cubic formula may be used):

Following Lee [37], replace a by $a + 1$ and check (using Mathematica) that

$$p(a + 1) = a^6 - 11a^4 + 35a^2 - 89 = b^3 - 11b^2 + 35b - 89,$$

a cubic in $b = a^2$ which can now be solved exactly for b , then for a , from which the original equation can be solved. It is seen that

$$\mu_0 = 3.8318740552 \dots$$

The other period 3-points may now be found since $c_1 = 1/2$, $c_2 = L_{\mu_0}(1/2) = \mu_0/4 = 0.95796 \dots$, and $c_3 = L_{\mu_0}^2(1/2) = \mu_0^2/4(1 - \mu_0/4) = 0.15248 \dots$

Example 2.6.4: The 3-Cycle when $\mu = 4$

When $\mu = 4$, $L_4(x) = 4x(1 - x)$. In this case we have

$$g_4(x) = 4096x^6 - 13312x^5 + 16640x^4 - 10048x^3 + 3024x^2 - 420x + 21,$$

and we can check using Mathematica (see also Lee [38])

$$\begin{aligned} g_4(x/4) &= x^6 - 13x^5 + 65x^4 - 157x^3 + 189x^2 - 105x + 21 \\ &= (x^3 - 7x^2 + 14x - 7)(x^3 - 6x^2 + 9x - 3), \end{aligned}$$

the product of two cubics. The solutions give a pair of 3-cycles, which we will show are given by the following theorem:

Theorem 2.6.5 *For the logistic map $L_4(x) = 4x(1 - x)$ we have:*

(i) *The 2-cycle is $\{\sin^2(\pi/5), \sin^2(2\pi/5)\}$*

(ii) *The 3-cycles are*

$$\{\sin^2(\pi/7), \sin^2(2\pi/7), \sin^2(3\pi/7)\} \quad \text{and} \quad \{\sin^2(\pi/9), \sin^2(2\pi/9), \sin^2(4\pi/9)\}.$$

Proof. Recall that the difference equation $x_{n+1} = 4x_n(1 - x_n)$, $n = 0, 1, 2, \dots$, has the solution

$$x_n = \sin^2(2^n \arcsin \sqrt{x_0}).$$

This was obtained by setting $x_n = \sin^2(\theta_n)$ for some $\theta_n \in (0, \pi/2]$, ($n = 1, 2, \dots$), so that

$$\sin^2(\theta_{n+1}) = 4\sin^2(\theta_n)(1 - \sin^2(\theta_n)) = 4\sin^2(\theta_n)\cos^2(\theta_n) = \sin^2(2\theta_n).$$

We can use this to show that $\sin^2(\theta_{n+1}) = \sin^2(4\theta_{n-1})$, so in general

$$x_n = \sin^2(\theta_n) = \sin^2(2^n \theta_0), \quad \text{where } \theta_0 = \arcsin(\sqrt{x_0}).$$

In particular we have

$$\theta_1 = \arcsin\left(\sqrt{\sin^2 2\theta_0}\right) = \begin{cases} 2\theta_0 & \text{for } 0 \leq \theta_0 \leq \pi/4 \\ \pi - 2\theta_0 & \text{for } \pi/4 \leq \theta_0 \leq \pi/2 \end{cases}$$

Applying this to the situation where we have a 2-cycle $\{c_0, c_1\}$, if $c_i = \sin^2(\theta_i)$, we get θ_0 is equal to $4\theta_0$, $\pi - 2\theta_0$, $2\pi - 4\theta_0$ or $\pi - 4\theta_0$. This gives $\theta = 0$ or $\theta = \pi/3$ (giving rise to the two fixed points) or $\theta_0 = \pi/5$, or $2\pi/5$ from which the result follows. A similar, but more complicated analysis gives the 3-cycles. $\square$

Remark 2.6.6 Using the above ideas, if we want to find the period n -points of L_4 , we solve the equation $L_4^n(x) = x$. When $x = \sin^2(\theta)$, this becomes

$$\sin^2(\theta) = \sin^2(2^n \theta).$$

This gives rise to the two equations

$$\pm\theta = 2^n\theta + 2k\pi, \quad \text{or} \quad \pm\theta = (2k+1)\pi - 2^n\theta, \quad \text{for some } k \in \mathbb{Z}.$$

This can be summarized as a single equation:

$$\pm\theta = 2^n\theta + k\pi \Rightarrow \theta = \frac{k\pi}{2^n \pm 1}, \quad n = 1, 2, 3, \dots, \quad k \in \mathbb{Z}$$

so that

$$\text{Per}_n(L_4) = \left\{ \sin^2\left(\frac{k\pi}{2^n - 1}\right) : 0 \leq k < 2^{n-1} \right\} \cup \left\{ \sin^2\left(\frac{k\pi}{2^n + 1}\right) : 0 < k \leq 2^{n-1} \right\}.$$

It follows that L_4 has points of all possible periods. We shall see that the set of all periodic points constitutes a “dense” set in $[0, 1]$ and each of these points is unstable.

Exercises 2.6

1. Recall the family of maps defined by $S_\mu(x) = \mu \sin(x)$ for $x \in [0, \pi]$ and $\mu \in [0, \pi]$. Use the Mathematica commands below together with the `ManipulatePlot` command to estimate the values of μ where periods two and three are born. Use

```
h[a_,x_] := a*Sin[x]
g[a_,x_] := h[a,h[a,x]]
k[a_,x_] := h[a,h[a,h[a,x]]]
```

```
Manipulate[Plot[{x,g[a,x]}, {x, 0, Pi}, PlotRange->{0,Pi},
AspectRatio->Automatic], {a,0,Pi}]
```

(Click on the + at the end of the `Manipulate` line to obtain the `Animation Controls`).

2. Modify the bifurcation diagram of the logistic maps to give a bifurcation diagram for the family S_μ , $0 \leq \mu \leq \pi$.
3. Do the same as in question 1 for the family $C_\mu(x) = \mu \cos(x)$, $x \in [-\pi, \pi]$ and $\mu \in [0, \pi]$.
4. Can you use the method of question 1 to estimate a value of μ for which S_μ has a superattracting 2-cycle, or 3-cycle?

5. Let $g_\mu(x) = \mu x \frac{(1-x)}{(1+x)}$, $\mu > 0$.

(a) Show that g_μ has a maximum at $x = \sqrt{2}-1$ and the maximum value is $\mu(3-2\sqrt{2})$.

(b) Deduce that g is a dynamical system on $[0, 1]$ for $0 \leq \mu \leq 3+2\sqrt{2}$ (i.e., $g_\mu([0, 1]) \subseteq [0, 1]$).

(c) Find the fixed points of g_μ for $\mu \geq 1$.

(d) Give conditions on μ for the fixed points of g_μ to be attracting.

(e) Use Mathematica to graph g_μ^2 and g_μ^3 , and estimate when a period 2-point is created.

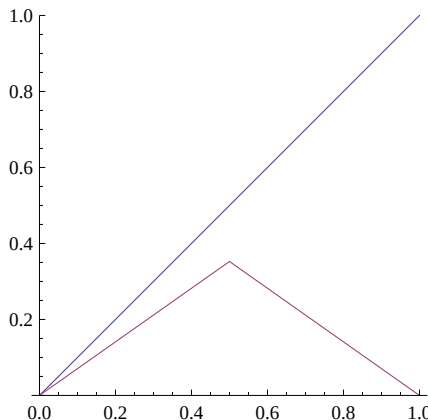
(f) Use Mathematica to give a bifurcation diagram for g_μ , for $0 \leq \mu \leq 3+2\sqrt{2}$.

2.7 The Tent Family T_μ .

We define another parameterized family in a piecewise linear manner: $T_\mu : [0, 1] \rightarrow [0, 1]$, $0 < \mu \leq 2$,

$$T_\mu(x) = \begin{cases} \mu x & \text{if } 0 \leq x \leq 1/2 \\ \mu(1 - x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

When $\mu = 2$ we get the familiar tent map $T(x)$ seen earlier. We now look at the parameter values $0 < \mu \leq 2$ (and later we shall see what happens when $\mu > 2$, and where $T_\mu : \mathbb{R} \rightarrow \mathbb{R}$). If $0 < \mu < 1$ we see that the only fixed point is $c = 0$. If $\mu = 1$, then all $c \in [0, 1/2]$ are fixed points and if $1 < \mu \leq 2$, then there are 2 fixed points. We look at each of these cases separately.



Case 2.7.1 $0 < \mu < 1$. We see that 0 is the only fixed point of T_μ and if $0 < x \leq 1/2$, then $0 \leq T_\mu(x) = \mu x < x$, and if $1/2 < x \leq 1$, then

$$0 \leq T_\mu(x) = \mu(1 - x) < 1 - x < \frac{1}{2} < x,$$

so in a similar manner to what we have seen previously, the sequence $T_\mu^n(x)$ is decreasing, bounded below by 0, so must converge to the fixed point 0. Thus $B_{T_\mu}(0) = [0, 1]$.

Case 2.7.2 $\mu = 1$ Clearly if $0 < x \leq 1/2$, then $T_1(x) = x$, so is a fixed point, and if $1/2 < x \leq 1$, then x is an eventual fixed point since

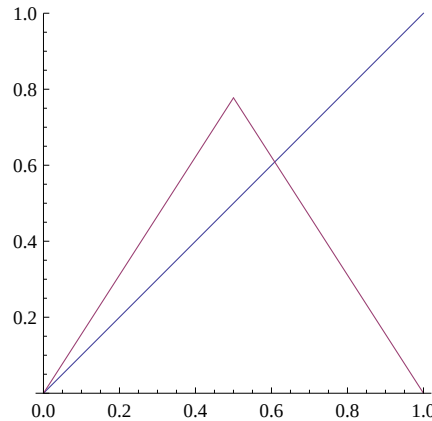
$$T_1^2(x) = T_1(1 - x) = x.$$

The fixed points are stable but not attracting.

Case 2.7.3 $1 < \mu < 2$. There is a bifurcation at $\mu = 1$ with a second fixed point c created, $1/2 < c < 1$. We solve $T_\mu(x) = x$ to find this point:

$$T_\mu(x) = \mu(1 - x) = x, \quad \text{so that} \quad c = \frac{\mu}{1 + \mu}.$$

Since $|T_\mu(x)| = \mu > 1$ in this case, the fixed point is repelling.



Case 2.7.4 $\mu = 2$. In this case $T_2 = T$, the familiar tent map. The fixed points 0 and $2/3$ are repelling. This time the range of T is all of $[0, 1]$. T has the effect of mapping the interval $[0, 1/2]$ onto all of $[0, 1]$ and then folding the interval $[1/2, 1]$ back over the interval $[0, 1]$. It is this stretching and folding that gives rise to the chaotic nature of T that we will see later, and is typical of many transformations of this type. We will examine the periodic points of T in the next section.

2.8 The 2-Cycles and 3-Cycles of the Tent Family

At some stage for $\mu \geq 1$ a 2-cycle is created. It is interesting to use Mathematica to do a dynamic iteration of T_μ to see how this happens (use the `ManipulatePlot` command described in Exercises 2.6, but with

```
h[x_,a_]:= a*Piecewise[{{x, x < 1/2}, {(1-x), 1/2 < x}}]
```

in place of `a*Sin[x]`). It can be checked that for $\mu > 1$, T_μ^2 is given by the formula

$$T_\mu^2(x) = \begin{cases} \mu^2 x & \text{if } 0 \leq x \leq \frac{1}{2\mu} \\ \mu(1 - \mu x) & \text{if } \frac{1}{2\mu} < x \leq \frac{1}{2} \\ \mu(1 - \mu + \mu x) & \text{if } \frac{1}{2} < x \leq 1 - \frac{1}{2\mu} \\ \mu^2(1 - x) & \text{if } 1 - \frac{1}{2\mu} < x \leq 1 \end{cases}$$

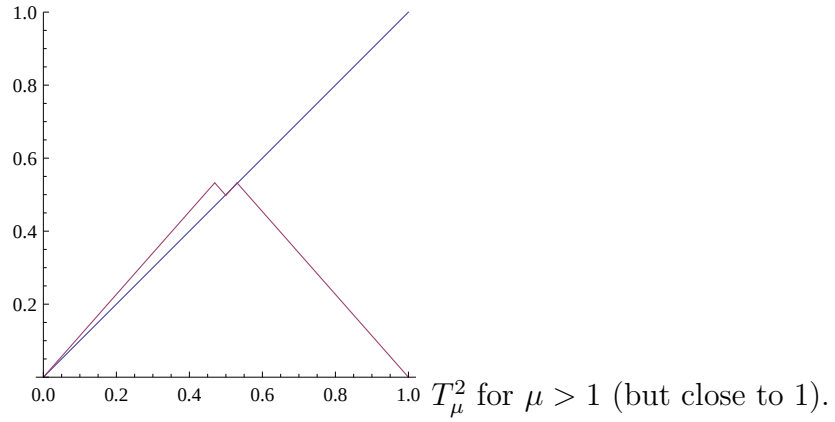
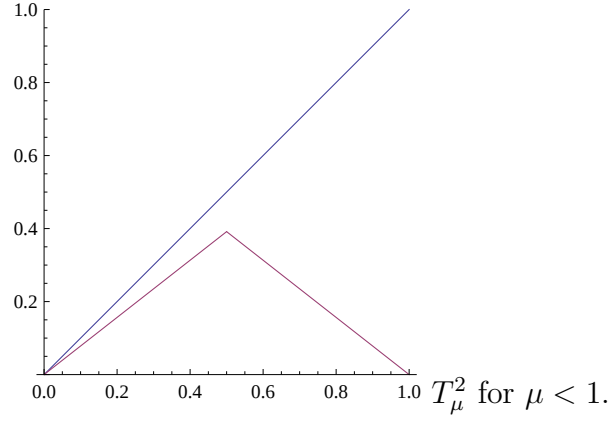
The 2-cycle is created when $T_\mu^2(1/2) = 1/2$, i.e., when

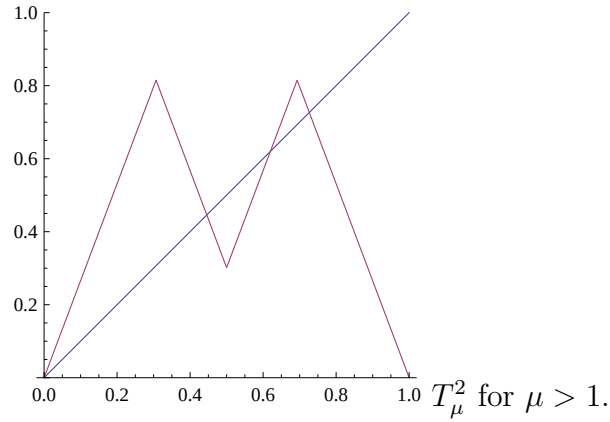
$$\mu(1 - \mu/2) = 1/2, \quad \text{or} \quad (\mu - 1)^2 = 0, \quad \mu = 1,$$

so the 2-cycle appears when $\mu > 1$. For $\mu \leq 1$, there are no period 2 points. The 2-cycle $\{c_1, c_2\}$ say, is unstable since

$$|(T_\mu^2)'(c_1)| = |T_\mu'(c_1)T_\mu'(c_2)| = \mu^2 > 1.$$

For example, if we solve $\mu(1 - \mu x) = x$, we get $c_1 = \frac{\mu}{1+\mu^2}$ and solving $\mu^2(1 - x) = x$ gives $c_2 = \frac{\mu^2}{1+\mu^2}$ as the period 2-points. Solving $\mu(1 - \mu + \mu x) = x$ gives $c = \frac{\mu}{1+\mu}$ as the (non-zero) fixed point.





We now look for the smallest value of μ for which there exists a 3-cycle. In a similar way to the situation for the 2-cycle, we look for where $T_\mu^3(1/2) = 1/2$ and we compute the value of the largest root of this equation. In general (see Heidel [29]) the smallest value of μ for which there exists a periodic orbit of period k is precisely the value of μ for which $1/2$ has period k . Using the formula for $T_\mu^2(1/2) = \mu(1 - \mu/2)$ above, then since $\mu(1 - \mu/2) \leq 1/2$ for all μ , we get

$$T_\mu^3(1/2) = \mu^2(1 - \mu/2) = 1/2 \quad \text{when} \quad \mu^3 - 2\mu^2 + 1 = 0,$$

or $(\mu - 1)(\mu^2 - \mu - 1) = 0$. Disregarding $\mu = 1$ and solving the quadratic gives $\mu = (1 + \sqrt{5})/2$ as the value of μ where period 3 first occurs.

In general it can be shown that for $k > 3$ odd, period k first occurs when μ is equal to the largest real root of the equation

$$\begin{aligned} & \mu^k - 2\mu^{k-1} + 2\mu^{k-3} - 2\mu^{k-4} + \dots - 2\mu + 1 \\ &= (\mu - 1)(\mu^{k-1} - \mu^{k-2} - \mu^{k-3} + \mu^{k-4} - \mu^{k-5} + \mu^{k-6} \dots + \mu - 1) = 0, \end{aligned}$$

so has $\mu - 1$ as a factor (see [29]). We will return to look at the tent family in Section 6.4.

Exercises 2.8

1. Find $\mu \in [0, 2]$ such that $c = 1/2$ is periodic of period 3 under the tent map T_μ (so that $\{1/2, \mu/2, \mu(1 - \mu/2)\}$ is a 3-cycle).
2. For $\mu > 1$, show that T_μ has no attracting periodic points.

3. If $L_\mu(x) = \mu x(1 - x)$ is such that $c = 1/2$ is n -periodic for some $n \in \mathbb{Z}^+$, prove that $1/2$ is an attracting periodic point. Is it necessarily a super-attracting periodic point?

4. Modify the bifurcation diagram of the logistic maps to give a bifurcation diagram for the tent family T_μ , $x \in [0, 1]$ and $\mu \in [0, 2]$.

Chapter 3. Sharkovsky's Theorem

3.1 Period 3 Implies Chaos. In 1975 in a paper entitled “Period three implies chaos”, Li and Yorke proved a remarkable theorem:

Theorem 3.1.1 *Let $f : X \rightarrow X$ be a continuous function defined on an interval $X \subseteq \mathbb{R}$. If $f(x)$ has a point of period three, then for any $k = 1, 2, 3, \dots$, there is a point having period k .*

This paper stirred considerable interest and shortly after it was pointed out that a Ukrainian mathematician by the name of Sharkovsky had in 1964 published a much more general theorem (in Russian) in a Ukrainian journal. His theorem was unknown in the west until the appearance of the Li-Yorke Theorem. To state his theorem we need to define a new ordering of the positive integers $\mathbb{Z}^+$. In the “Sharkovsky ordering”, 3 is the largest number, followed by 5 then 7 (all of the odd integers), then $2 \cdot 3$, $2 \cdot 5$, (2 times the odd integers), then 2^2 times the odd integers etc., finishing off with powers of 2 in descending order:

$$3 \triangleright 5 \triangleright 7 \triangleright \dots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \dots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \dots \triangleright 2^n \cdot 3 \triangleright 2^n \cdot 5 \triangleright \dots \triangleright 2^n \triangleright 2^{n-1} \triangleright \dots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1.$$

Sharkovsky's Theorem says that if a continuous map has a point of period k , then it has points of all periods less than k in the Sharkovsky ordering. The converse is also true in the sense that for each $k \in \mathbb{Z}^+$ there is a map having points of period k , but no points of period larger than k in the Sharkovsky ordering.

Theorem 3.1.2 (Sharkovsky's Theorem, 1964.) *Let $f : X \rightarrow X$ be a continuous map on an interval X (where X may be any bounded or unbounded subinterval of $\mathbb{R}$). If f has a point of period k , then it has points of period r for all $r \in \mathbb{Z}^+$ with $k \triangleright r$.*

For example, this theorem tells us that if f has a 4-cycle, then it also has a 2-cycle and a 1-cycle (fixed point). If f has a 3-cycle, then f has all other possible cycles. If f has a 6-cycle, then since $6 = 2 \cdot 3$, f will have $2 \cdot 5$, $2 \cdot 7$, $\dots$, $2^2 \cdot 3$, $2^2 \cdot 5$, $\dots$ -cycles, etc.

We omit the proof of this theorem, but shall prove the case where $k = 3$ (often known as the Li-Yorke Theorem because of their independent proof of this result in 1975). James Yorke is a professor at UMD, College Park and Li was his graduate student. A few years ago, Yorke and Benoit Mandelbrot were awarded the Japan Prize (the Japanese equivalent of the Nobel Prize) for their work in Dynamical Systems and Fractals (Mandelbrot is regarded as the “father” of fractals, and we will review some

of his work later). In order to prove this result we shall need some preliminary lemmas. The first was proved in Chapter 1 (Theorem 1.2.9).

Lemma 3.1.3 *Let $f : I \rightarrow \mathbb{R}$ be a continuous map, I an interval with $J = f(I) \supseteq I$, then $f(x)$ has a fixed point in I .*

Lemma 3.1.4 *Let $f : I \rightarrow \mathbb{R}$ be a continuous map. If $J \subseteq f(I)$ is a closed bounded interval, then there exists a closed bounded interval $K \subseteq I$ with $f(K) = J$.*

Proof. Write $J = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$.

There are $r, s \in I$ with $f(r) = a$ and $f(s) = b$. Set

$$\beta = y, \quad \text{where } y \in I, \quad |y - r| \text{ is a minimum, and } f(y) = b,$$

(β exists from continuity and $f(\beta) = b$: β is the point of I that is closest to r with the property that $f(\beta) = b$). Similarly, set

$$\alpha = x, \quad \text{where } x \text{ lies between } \beta \text{ and } r, \quad |x - \beta| \text{ is a minimum, and } f(x) = a$$

(possibly equal to r), so as before $f(\alpha) = a$. In this way, there is no $c \in (\alpha, \beta)$ where $f(c) = a$, or $f(c) = b$. Then the closed interval K (either $K = [\alpha, \beta]$ or $K = [\beta, \alpha]$), has the property $f(K) = [a, b]$, since firstly, $f(\alpha) = a$, $f(\beta) = b$, so by the Intermediate Value Theorem $[a, b] \subseteq f(K)$. On the other hand, if $w \in f(K)$, then $w = f(z)$ for some z between α and β . This must give $f(z) \in [a, b]$ since otherwise, another application of the Intermediate Value Theorem would contradict the choices of α and β . $\square$

3.1.5 Proof of Sharkovsky's Theorem for $k = 3$

We are assuming that f has a point of period 3, so there is a 3-cycle $\{a, b, c\}$ with $f(a) = b$, $f(b) = c$ and $f(c) = a$. We assume that $a < b < c$ (the other case with $a < c < b$ may be treated similarly).

We give the idea of the proof by showing why there must be points of period one, two and four. Let

$$[a, b] = L_0 \quad \text{and} \quad [b, c] = L_1.$$

Observe that

$$f(L_0) \supseteq L_1 \quad \text{and} \quad f(L_1) \supseteq L_0 \cup L_1.$$

Case 1. f has a fixed point.

Since

$$f(L_1) \supseteq L_0 \cup L_1 \supseteq L_1,$$

Lemma 3.1.3 implies that f has a fixed point in L_1 .

Case 2. f has a point of period 2.

This time we use

$$f(L_1) \supseteq L_0 \cup L_1 \supseteq L_0,$$

so by Lemma 3.1.4 there is a set $B \subseteq L_1$ such that $f(B) = L_0$. We then have

$$f^2(B) = f(L_0) \supseteq L_1 \supseteq B,$$

so by Lemma 3.1.3, B contains a fixed point c of f^2 . c is a period 2-point of f (and not a fixed point of f) because

$$f(c) \in L_0 \quad \text{and} \quad c \in L_1, \quad \text{so} \quad f(c) \neq c.$$

Case 3. f has a point of period 4.

The above two constructions do not illustrate the general method, but the following construction is easily generalized to any number of fixed points greater than 3. Our aim is to show that there is a subset B of L_1 which is mapped first by f into L_1 , then into L_1 again, then onto L_0 and then onto L_1 , so that $f^4(B) \supseteq B$. Thus f^4 has a fixed point c in B , which cannot be a point of lesser period because $f(c) \in L_1$, $f^2(c) \in L_1$, $f^3(c) \in L_0$ and $f^4(c) \in B$ (so cannot have $f(c) = c$, $f^2(c) = c$ or $f^3(c) = c$).

It is useful to think of 5 copies of $L_0 \cup L_1$ with f mapping the first to the second etc. as shown:

We find sets B_1 , B_2 and B_3 as follows:

$$f(L_0) \supseteq L_1, \quad f(L_1) \supseteq L_0 \cup L_1,$$

so there exists $B_1 \subseteq L_1$ such that $f(B_1) = L_0$.

There exists $B_2 \subseteq L_1$ such that $f(B_2) = B_1$ and there exists $B_3 \subseteq L_1$ such that $f(B_3) = B_2$. Set $B = B_3$, then

$$f^2(B_3) = f(B_2) = B_1, \quad \text{and so} \quad f^3(B) = L_0, \quad f^4(B) \supseteq L_1 \supseteq B_3.$$

In other words $f^4(B) \supseteq B$, so there exists $c \in B$, a fixed point of f^4 , which is not a point of period 3 or less, so must be a point of period 4. $\square$

In general if a function has points of period 4 the most we can deduce is that there are points of period 2 and fixed points. However, the following is true:

Proposition 3.1.6 *If $f : I \rightarrow I$ is continuous on an interval I with*

$$f(a) = b, \quad f(b) = c, \quad f(c) = d, \quad f(d) = a, \quad a < b < c < d,$$

then $f(x)$ has a point of period 3, so also has points of all other periods.

Proof. We may assume that

$$f[a, b] = [b, c], \quad f[b, c] = [c, d], \quad f[c, d] = [a, d].$$

In particular, there exists $B_1 \subseteq [c, d]$ with $f(B_1) = [c, d]$. There exists $B_2 \subseteq [c, d]$ with $f(B_2) = [b, c]$.

Take $K_1 \subseteq B_1$ with $f(K_1) = B_2$, then

$$f^3(K_1) = f^2(B_2) = f[b, c] = [c, d] \supseteq K_1,$$

so f^3 has a fixed point in K_1 which is not a fixed point of $f(x)$. $\square$

The above proofs can be summarized with the following type of result:

Proposition 3.1.7 *Let $f : I \rightarrow I$ be a continuous map, and let I_1 and I_2 be two closed subintervals of I with at most one point in common. If $f(I_1) \supset I_2$ and $f(I_2) \supset I_1 \cup I_2$, then f has a 3-cycle.*

Proof. Exercise: Use the ideas of the proof of 3.1.5. $\square$

3.2 Converse of Sharkovsky's Theorem

As we mentioned, for each $m \in \mathbb{Z}^+$ in the Sharkovsky ordering of $\mathbb{Z}^+$, Sharkovsky showed that there is a continuous map $f : I \rightarrow I$ (I an interval), such that $f(x)$ has a point of period m , but no point of period k for $k \succ m$. The following were shown (as usual, I is either the real line or an interval):

Theorem 3.2.1 *For every $k \in \mathbb{Z}^+$, there exists a continuous map $f : I \rightarrow I$ that has a k -cycle, but has no cycles of period n for any n appearing before k in the Sharkovsky ordering.*

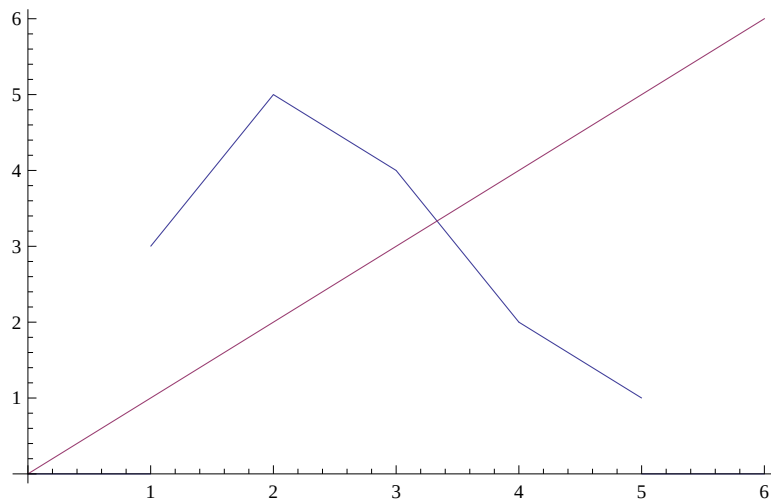
Theorem 3.2.2 *There exists a continuous map $f : I \rightarrow I$ that has a 2^n -cycle, for every $n \in \mathbb{Z}^+$ and has no other cycles of any other period.*

Strictly speaking, Sharkovsky's Theorem is really the combination of Theorem 3.1.2, Theorem 3.2.1 and Theorem 3.2.2 (see [44]), and sometimes the latter two theorems are referred to as the converse of Sharkovsky's Theorem (see [23]). We look at some particular cases of this:

Example 3.2.3 Define a function $f : [1, 5] \rightarrow [1, 5]$ as shown by the graph below (so

$$f(1) = 3, \quad f(2) = 5, \quad f(3) = 4, \quad f(4) = 2, \quad f(5) = 1,$$

with $f(x)$ piecewise linear between these points). Then f has a point of period 5, but no point of period 3.



Proof. Clearly no point from the set $\{1, 2, 3, 4, 5\}$ is of period 3, but this set is a 5-cycle. Also, Theorem 1.2.7 tells us that $f(x)$ has a fixed point c , so c is a fixed point for f^3 . We shall show that f^3 has no other fixed points. Suppose to the contrary that

f^3 has another fixed point α . Now we can check that:

$$f^3[1, 2] = [2, 5], \quad f^3[2, 3] = [3, 5], \quad f^3[4, 5] = [1, 4],$$

so f^3 cannot have a fixed point in the intervals $[1, 2]$, $[2, 3]$ or $[4, 5]$, so α must lie in the interval $[3, 4]$. In fact $f^3[3, 4] = [1, 5] \supseteq [3, 4]$ and we show that f^3 cannot have another fixed point in $[3, 4]$.

If $\alpha \in [3, 4]$, then $f(\alpha) \in [2, 4]$, so either $f(\alpha) \in [2, 3]$ or $f(\alpha) \in [3, 4]$. If the former holds, then $f^2(\alpha) \in [4, 5]$ and $f^3(\alpha) \in [1, 2]$ which is impossible as we have to have $f^3(\alpha) = \alpha \in [3, 4]$.

Thus we must have $f(\alpha) \in [3, 4]$, so that $f^2(\alpha) \in [2, 4]$. Again there are two possibilities: if $f^2(\alpha) \in [2, 3]$ then $f^3(\alpha) \in [4, 5]$, another contradiction, so that $f^2(\alpha) \in [3, 4]$.

We have shown that the orbit of α : $\{\alpha, f(\alpha), f^2(\alpha)\}$ is contained in the interval $[3, 4]$. On the interval $[3, 4]$ we can check that $f(x)$ is given by the straight line formula

$$f(x) = 10 - 2x, \quad \text{and} \quad f(10/3) = 10/3,$$

so $c = 10/3$ is the unique fixed point of f . Also

$$f^2(x) = -10 + 4x, \quad f^3(x) = 30 - 8x,$$

also with the unique fixed point $x = 10/3$. It follows that f cannot have any points of period 3. $\square$

Remark 3.2.2 The above can be directly generalized to give a map having points of period $2n + 1$, but no points of period $2n - 1$, $n = 2, 3, \dots$.

Exercises 3.2

1. Use the ideas of Chapter 3 to show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and has a 2-cycle $\{a, b\}$, then f has fixed point.
2. Show that the map $f(x) = (x - 1/x)/2$, $x \neq 0$, has no fixed points but it has period 2-points. Find the 2-cycle and by looking at the graph of $f^3(x)$, check to see whether it has a 3-cycle. Why does this not contradict Sharkovsky's Theorem?

3. A map $f : [1, 7] \rightarrow [1, 7]$ is defined so that $f(1) = 4, f(2) = 7, f(3) = 6, f(4) = 5, f(5) = 3, f(6) = 2, f(7) = 1$, and the corresponding points are joined so the map is piecewise linear. Show that f has a 7-cycle but no 5-cycle.
4. If $F_\lambda(x) = 1 - \lambda x^2$ for $x \in \mathbb{R}$, show
- (i) F_λ has fixed points for $\lambda \geq -1/4$.
 - (ii) F_λ has a 2-cycle for $\lambda > 3/4$.
 - (iii) The 2-cycle is attracting for $3/4 < \lambda < 5/4$.
5. Show that an increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ cannot have a 3-cycle. Can it have a 2-cycle? Answer the same questions when f is decreasing.

Chapter 4. Metric Spaces

In order to generalize some of the ideas we have met so far, it is convenient to introduce the idea of a *metric space* (see [60] for example). This is simply a pair (X, d) where X is a set and d is a distance defined on the set, called a *metric*. d must satisfy certain natural properties that one would expect of a distance function:

4.1 Basic Properties of Metric Spaces

Definition 4.1.1 Let X be a set with $x, y, z \in X$. A metric on X is a function $d : X \times X \rightarrow \mathbb{R}^+$ satisfying:

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, y) \leq d(x, z) + d(z, y)$ (the *triangle inequality*).

We see that a metric on X is just a “distance function” ($d(x, y)$ = the distance from x to y) on X with certain natural properties).

Definition 4.1.2 The pair (X, d) is called a *metric space*, i.e., a metric space is just a set X with a distance function d satisfying (1)-(4) above.

Examples 4.1.3 1. $X = \mathbb{R}$ with $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ is a metric space. In a similar way, $X = [0, 1]$ with the same distance function is a metric space.

2. $X = \mathbb{R}^2$ with the usual distance in the plane defined by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ is a metric space. More generally $\mathbb{R}^n$ with its usual distance function and $\mathbb{C}$ the set of all complex numbers with $d(z, w) = |z - w|$, $z, w \in \mathbb{C}$ are metric spaces.

3. Let $X = S^1$ where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane. The natural metric on S^1 is given by the (shortest) distance around the circle between the two points in question.

4. If $X =$ any set and $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$, for $x, y \in X$, then d is a metric on X , called the *discrete metric*.

5. Let $X = \Sigma = \{(s_1, s_2, s_3, \dots) : s_i = 0 \text{ or } 1\}$, so that the members of Σ are infinite sequences of zeros and ones. We define a distance on Σ as follows: If

$$\omega_1 = (s_1, s_2, s_3, \dots), \quad \omega_2 = (t_1, t_2, t_3, \dots),$$

are members of Σ , then set

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{|s_n - t_n|}{2^n},$$

then it is easy to check that d is a metric on Σ , so (Σ, d) is a metric space.

For example, suppose that $\omega_1 = (1, 1, 1, 1, \dots)$ and $\omega_2 = (1, 0, 1, 0, 1, \dots)$, then

$$d(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{3}.$$

With this metric, points with coordinates which differ for small n are further apart than those with coordinates that differ for large values of n . For example, the distance between $(1, 0, 1, 1, 1, \dots)$ and $(1, 1, 1, 1, \dots)$ is $1/4$, whereas the distance between $(1, 1, 1, 0, 1, 1, 1, \dots)$ and $(1, 1, 1, 1, 1, 1, \dots)$ is $1/16$.

Definition 4.1.4 If (X, d) is a metric space and $a \in X$, $\epsilon > 0$, then

$$B_\epsilon(a) = \{x \in X : d(a, x) < \epsilon\},$$

is called the *open ball* centered on a with radius ϵ .

Examples 4.1.5 If $X = \mathbb{R}$ as in the above example, then the open balls are intervals of the form $(a - \epsilon, a + \epsilon)$. If $X = \mathbb{R}^2$, and $a = (a_1, a_2)$ is a point of $\mathbb{R}^2$, the points of $B_\epsilon(a)$ are those interior to the circle centered on (a_1, a_2) and of radius ϵ (the boundary of the circle is not included).

Definition 4.1.6 (i) A set $A \subseteq X$ is said to be *open* if for each $a \in A$ there exists $\epsilon > 0$ satisfying $B_\epsilon(a) \subseteq A$ (i.e., every point of A can be surrounded by an open ball entirely contained in A).

(ii) The set $A \subseteq X$ is *closed* if its complement $X - A$ (sometimes written A^c) is open.

(iii) A point $a \in X$ is a *limit point* of a set $A \subseteq X$ if every ball $B_\epsilon(a)$ contains a point of A other than a .

Examples 4.1.7 1. The open intervals (a, b) in $\mathbb{R}$ are open sets and any union of open intervals is open. The closed intervals such as $[a, b]$, $[a, \infty)$ are closed sets. Sets such as $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are neither open nor closed.

2. Any open ball $B_\epsilon(a)$ in a metric space X is an open set. The *closed ball*:

$$\overline{B}_\epsilon(a) = \{x \in X : d(a, x) \leq \epsilon\},$$

will be a closed set. The empty set $\emptyset$ and the whole space X are both open and closed.

3. The union of open sets is open and the intersection of closed sets is closed.

Proof. If $A = \cup_{\alpha \in \Lambda} A_\alpha$ is a union of open sets, let $a \in A$, then by definition, $a \in A_\alpha$ for some $\alpha \in \Lambda$. Since A_α is open, there exists $\delta > 0$ with $B_\delta(a) \subseteq A_\alpha$. It follows that $B_\delta(a) \subseteq A$, so A is open.

If $C = \cap_{\alpha \in \Lambda} C_\alpha$ is an intersection of closed sets, then each $X - C_\alpha$ is open. But DeMorgan's laws tell us that

$$\cup_{\alpha \in \Lambda} (X - C_\alpha) = X - \cap_{\alpha \in \Lambda} C_\alpha,$$

so that $\cap_{\alpha \in \Lambda} C_\alpha$ is closed. □

4. In $\mathbb{R}$, 0 is a limit point of both the interval $(0, 1)$ and the set $\{1, 1/2, 1/3, \dots, 1/n, \dots\}$. This is because every open ball centered on 0 contains points of the set in question, other than 0.

5. If X is given the discrete metric and $0 < \epsilon < 1$ then

$$B_\epsilon(a) = \{a\}, \quad \text{for all } a \in X.$$

6. If (X, d) is a metric space and A is a subset of X , then we may regard (A, d) as a metric space. For example if $A = [0, 1]$ the subset of $\mathbb{R}$ with the usual metric, then in A , $B_\epsilon(0) = [0, \epsilon]$ is the open ball centered on 0.

4.2 Dense Sets

In order to define the notion of chaos for one-dimensional maps, we need various topological notions such as denseness and transitivity.

Definition 4.2.1 The *closure* of a set A in a metric space X is defined to be

$$\overline{A} = A \cup \{ \text{the limit points of } A \}.$$

Proposition 4.2.2 $\overline{A}$ is the smallest closed set containing A , (i.e., if B is another closed set containing A then $\overline{A} \subseteq B$).

Proof. Clearly $A \subset \overline{A}$. To see that $\overline{A}$ is a closed set, let $a \in X - \overline{A}$. Then a is not a limit point of A so there exists $\delta > 0$ with $B_\delta(a) \cap A = \emptyset$.

We claim $B_\delta(a) \cap \overline{A} = \emptyset$, for if x belongs to this set then $x \notin A$ but x is a limit point of A . Also $x \in B_\delta(a)$, an open set, so there exists $\epsilon > 0$ with $B_\epsilon(x) \subseteq B_\delta(a)$. Hence

$$B_\epsilon(x) \cap A \subseteq B_\delta(a) \cap A = \emptyset,$$

contradicting the fact that x is a limit point of A .

It now suffices to show that if B is any other closed set containing A , then $\overline{A} \subseteq B$. Let $x \in X - B$, an open set, then

$$B_\delta(x) \subset X - B \subseteq X - A \quad \text{for some } \delta > 0,$$

and in particular $B_\delta(x) \cap A = \emptyset$. This says that x is not a limit point of A , so $x \in X - \overline{A}$. This shows that $X - B \subseteq X - \overline{A}$, so $\overline{A} \subseteq B$. $\square$

Definition 4.2.3 The set $A \subseteq X$ is *dense* in X if $\overline{A} = X$.

Examples 4.2.4 1. Let $I \subseteq \mathbb{R}$ be an interval. $A \subseteq I$ is *dense* in I if for any open interval U contained in I we have $U \cap A \neq \emptyset$. This is because every $x \in I$ is a limit point of A : if $x \in I$, $B_\delta(x) \cap A \neq \emptyset$ for every $\delta > 0$.

Equivalently A is dense in I if for any $x \in I$ and any $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains a point of A .

Intuitively, the points of A are spread uniformly over the interval I in such a way that every subinterval of I (no matter how small) contains some points of A .

2. The set $\mathbb{Q}$ of all rational numbers is dense in $\mathbb{R}$. The set $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$.

Proof. We show that the set $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$. Let $x \in (0, 1)$, from the definition it suffices to find a rational number y arbitrarily close to x , i.e., satisfying $|x - y| < \delta$, for some arbitrary number $\delta > 0$. Suppose that x has a decimal expansion

$$x = \sum_{n=1}^{\infty} \frac{d_n}{10^n} = .d_1d_2d_3\dots, \quad \text{where } d_n \in \{0, 1, 2, \dots, 9\}.$$

Choose $m \in \mathbb{Z}^+$ so large that $10^{-m} < \delta$ and set $y = \sum_{n=0}^m \frac{d_n}{10^n} \in \mathbb{Q}$, then

$$|x - y| = |\cdot d_{m+1} d_{m+2} \dots| = \sum_{n=m+1}^{\infty} \frac{d_n}{10^n} \leq \sum_{n=m+1}^{\infty} \frac{9}{10^n} \leq \frac{1}{10^m} < \delta.$$

□

3. The set $\mathbb{R} - \mathbb{Q}$ of all irrational numbers is dense in $\mathbb{R}$.
4. For intervals in $\mathbb{R}$, $\overline{(a, b)} = [a, b]$. If $A = \{1, 1/2, 1/3, \dots\}$, then $\overline{A} = A \cup \{0\}$.
5. A proper subset of a metric space X can be both open and dense. For example, in $\mathbb{R}$, the set $E = \mathbb{R} - \{\sqrt{2}\}$ is both open and dense, with the property that $\mathbb{Q} \subset E \neq \mathbb{R}$.

Definition 4.2.5 If (X, d) is a metric space, then we say $\lim_{n \rightarrow \infty} x_n = a$ if for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $d(a, x_n) < \epsilon$ for all $n > N$.

Theorem 4.2.6 *The following are equivalent for a metric space (X, d) :*

- (i) *The set A is dense in X .*
- (ii) *Let $\epsilon > 0$, then for all $x \in X$ there exists $a \in A$ such that $d(a, x) < \epsilon$.*
- (iii) *For all $x \in X$, there is a sequence $\{a_n\} \subset A$ with $\lim_{n \rightarrow \infty} a_n = x$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that A is dense in X and let $x \in X$, then either $x \in A$ (so (ii) holds) or x is a limit point of A :

i.e., any ball $B_\epsilon(x)$ contains points of A , so there exists $a \in A$ with $d(x, a) < \epsilon$.

(ii) $\Rightarrow$ (iii). Let $x \in X$ and $n \in \mathbb{Z}^+$, then there exists $a_n \in A$ with $d(x, a_n) < 1/n$ ($n = 1, 2, \dots$), so that as $n \rightarrow \infty$, $d(x, a_n) \rightarrow 0$.

i.e., given any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that

$$n > N \Rightarrow d(x, a_n) < \epsilon, \quad \text{or} \quad \lim_{n \rightarrow \infty} a_n = x.$$

(iii) $\Rightarrow$ (i). We must show that $\overline{A} = X$. Let $x \in X$, then there exists a sequence $\{a_n\} \subset A$ with $\lim_{n \rightarrow \infty} a_n = x$, i.e., x is a limit point of A since $B_\delta(x) \cap A \neq \emptyset$ for each $\delta > 0$, so $\overline{A} = X$. □

Example 4.2.7 1. For a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ the set $\text{Per}_1(f) = \text{Fix}(f)$ of all fixed points is a closed set. The same is true for continuous functions on a metric space (see below).

Proof. Let $x \in X$ and suppose that x_n is a sequence of points in $\text{Fix}(f)$ with $x_n \rightarrow x$. The continuity of f implies $\lim_{n \rightarrow \infty} f(x_n) = f(x)$, or

$$x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f(x_n) = f(x),$$

so x is also fixed point and $\text{Fix}(f)$ is closed.

2. If $f(x) = x \sin(1/x)$, then $f(x) = x$ when $x = 0$ and when $x = x_k = \frac{2}{\pi(1+4k)}$, $k \in \mathbb{Z}$. Note that $\lim_{k \rightarrow \infty} x_k = 0$. Also each x_k is a non-hyperbolic fixed point and $f'(0)$ is not defined. The set $\text{Fix}(f) = \{x_k : k \in \mathbb{Z}\} \cup \{0\}$ is clearly a closed set. This function has many eventually fixed points which are easy to find. The fixed point $x = 0$ is stable but not attracting since there are other fixed points arbitrarily close to it.

Exercises 4.1 - 4.2

1. (i) Prove that every open ball $B_\epsilon(a)$ in a metric space (X, d) is an open set and that every finite subset is a closed set.

(ii) Show that in general, the closed ball $\{x \in X : d(a, x) \leq \epsilon\}$ need not be equal to the closure of the open ball $B_\epsilon(a)$ (Hint: Consider the two point space $X = \{a, b\}$ with metric $d(a, b) = 1$).

2. For $x, y \in \mathbb{R}$, distances d_1, d_2, d_3, d_4, d_5 are defined by:

$$d_1(x, y) = (x - y)^2, \quad d_2(x, y) = \sqrt{x - y}, \quad d_3(x, y) = |x^2 - y^2|,$$

$$d_4(x, y) = |x - 2y|, \quad d_5(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Which of these (if any) defines a metric on $\mathbb{R}$? (Hint: For d_5 , consider the properties of the map $f(x) = x/(1+x)$).

3. Show that for $x = (x_1, x_2), y = (y_1, y_2)$, points in $\mathbb{R}^2$, the following define metrics on $\mathbb{R}^2$:

$$d'(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}, \quad d''(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

What do the open balls $B_1(a)$, where $a = (a_1, a_2)$, look like in each case?

4. Show that the intersection of a finite number of open sets $A_1, A_2, \dots, A_n$, in a metric space (X, d) is an open set. Show, by considering the intervals $(-1/n, 1/n)$ ($n \in \mathbb{Z}^+$), in $\mathbb{R}$ that the intersection of infinitely many open sets need not be open.

5. Denote by Σ the metric space of all sequences of 0's and 1's

$$\Sigma = \{\omega = (s_1, s_2, s_3, \dots) : s_i = 0 \text{ or } 1\},$$

with metric

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k}$$

when $\omega_1 = (s_1, s_2, \dots)$ and $\omega_2 = (t_1, t_2, \dots)$. Find $d(\omega_1, \omega_2)$ if:

- (i) $\omega_1 = (0, 1, 1, 1, 1, \dots)$ and $\omega_2 = (1, 0, 1, 1, 1, \dots)$,
- (ii) $\omega_1 = (0, 1, 0, 1, 0, \dots)$ and $\omega_2 = (1, 0, 1, 0, 1, \dots)$.

6. Denote by $C[a, b]$ the set of all continuous functions $f : [a, b] \rightarrow \mathbb{R}$. If the distance between two such continuous functions is given by:

$$d_1(f, g) = \max_{x \in [a, b]} |f(x) - g(x)| \quad \text{or} \quad d_2(f, g) = \int_a^b |f(x) - g(x)| dx,$$

then show that both d_1 and d_2 define metrics on $C[a, b]$.

4.3 Functions Between Metric Space

Given two different metric spaces (X, d_1) and (Y, d_2) , we may need to consider functions between them. Here the notion of continuity is often important:

Definition 4.3.1 A function $f : X \rightarrow Y$ between two metric spaces (X, d_1) and (Y, d_2) is said to be continuous at $a \in X$ if given any $\epsilon > 0$ there exists $\delta > 0$ such

that if $x \in X$ then

$$d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \epsilon.$$

Proposition 4.3.2 *The following are equivalent for $f : X \rightarrow Y$ between metric spaces (X, d_1) and (Y, d_2) :*

- (i) f is continuous at $x = a$.
- (ii) Given any ϵ -ball $B_\epsilon(f(a))$ centered on $f(a)$ there exists a δ -ball $B_\delta(a)$ centered on a such that

$$f(B_\delta(a)) \subset B_\epsilon(f(a)).$$

- (iii) Given any open set V containing $f(a)$ there exists an open set U containing a such that

$$f(U) \subseteq V.$$

Proof. Since the equivalence of (i) and (ii) is a fairly straightforward restatement of the definition of continuity, we will only show the equivalence of (i) and (iii).

(i) $\Rightarrow$ (iii) Let $V \subset Y$ be open with $f(a) \in V$. Then there exists $\epsilon > 0$ with $B_\epsilon(f(a)) \subseteq V$ (since V is open). Since f is continuous at a , there exists $\delta > 0$ with

$$f(B_\delta(a)) \subseteq B_\epsilon(f(a)) \subseteq V,$$

so set $U = B_\delta(a)$ and the result follows.

(iii) $\Rightarrow$ (i) $B_\epsilon(f(a))$ is open in Y and contains $f(a)$, so by hypothesis there exists U open in X , containing a such that

$$f(U) \subseteq B_\epsilon(f(a)).$$

Since U is open, we can find $\delta > 0$ such that $B_\delta(a) \subseteq U$. Thus

$$f(B_\delta(a)) \subseteq f(U) \subseteq B_\epsilon(f(a)),$$

so that f is continuous at a □

Definition 4.3.3 A function $f : X \rightarrow Y$ is *continuous* if it is continuous at every $a \in X$.

Theorem 4.3.4 *The following are equivalent for a function $f : X \rightarrow Y$ between metric spaces:*

(i) $f : X \rightarrow Y$ is continuous.

(ii) $f^{-1}(V)$ is open in X whenever V is open in Y ($f^{-1}(V) = \{x \in X : f(x) \in V\}$ is defined even if f is not invertible).

Proof. (i) $\Rightarrow$ (ii) Let V be open in Y , then we may assume $f^{-1}(V) \neq \emptyset$, so let $a \in f^{-1}(V)$, then $f(a) \in V$. Since V is open there exists $B_\epsilon(f(a)) \subset V$. Now f is continuous at a , so there exists $\delta > 0$ such that

$$f(B_\delta(a)) \subseteq B_\epsilon(f(a)), \quad \text{so} \quad B_\delta(a) \subseteq f^{-1}B_\epsilon(f(a)) \subseteq f^{-1}(V),$$

so $f^{-1}(V)$ is open in X .

(ii) $\Rightarrow$ (i) If $f^{-1}(V)$ is open in X whenever V is open in Y , let $a \in X$, then $f(a) \in B_\epsilon(f(a))$, an open set in Y ($\epsilon > 0$).

By hypothesis, $f^{-1}(B_\epsilon(f(a)))$ is open in X and since $a \in f^{-1}(B_\epsilon(f(a)))$, there exists $\delta > 0$ such that

$$B_\delta(a) \subseteq f^{-1}B_\epsilon(f(a)) \quad \text{or} \quad f(B_\delta(a)) \subseteq B_\epsilon(f(a)),$$

so that f is continuous at a . □

As an application of this theorem we show that the basin of attraction $B_f(p)$, of a fixed point p for a continuous map $f : X \rightarrow X$, X a metric space, is an open set. Let p be an attracting fixed point for f . In this context it means that there exists $\epsilon > 0$ such that if $x \in B_\epsilon(p)$, the open ball of radius ϵ centered on p , then $f^n(x) \rightarrow p$ as $n \rightarrow \infty$. Recall that the basin of attraction of p is the set

$$B_f(p) = \{x \in X : f^n(x) \rightarrow p \text{ as } n \rightarrow \infty\}.$$

We saw that for $0 < \mu < 1$, 0 is an attracting fixed point of the logistic map $L_\mu(x) = \mu x(1 - x)$ with basin of attraction $[0, 1]$. In this case the metric space is $X = [0, 1]$, so the basin of attraction is an open set namely the whole space (the fixed point 0 is globally attracting). For continuous maps $f : \mathbb{R} \rightarrow \mathbb{R}$ having an asymptotically stable fixed point p , the basin of attraction is an open set, which in this case is just the union of open intervals. The largest such open interval to which p belongs is called the *immediate basin of attraction* of p under f .

To say that $A \subseteq X$ is *invariant* under f means that $f(x) \in A$ for all $x \in A$.

Theorem 4.3.5 *If $f : X \rightarrow X$ is a continuous map of a metric space X and p is an attracting fixed point of f , then $B_f(p)$, the basin of attraction of f , is an invariant open set.*

Proof. If $x \in B_f(p)$, then $f^n(x) \rightarrow p$ as $n \rightarrow \infty$, so that $f^n(f(x)) \rightarrow f(p) = p$ as $n \rightarrow \infty$, so $f(x) \in B_f(p)$ and $B_f(p)$ is an invariant set.

Since p is an attracting fixed point there exists $\epsilon > 0$ such that if $x \in B_\epsilon(p)$, $f^n(x) \rightarrow p$ as $n \rightarrow \infty$. Since f is a continuous function the set $f^{-1}(B_\epsilon(p))$ is open (from Theorem 4.3.4). We claim that

$$B_f(p) = \cup_{n=1}^{\infty} f^{-n}(B_\epsilon(p)),$$

from which the result follows since the union of open sets is open.

Let $x \in \cup_{n=1}^{\infty} f^{-n}(B_\epsilon(p))$, then $x \in f^{-n}(B_\epsilon(p))$ for some $n \in \mathbb{Z}^+$, or $f^n(x) \in B_\epsilon(p)$, so that clearly $x \in B_f(p)$.

On the other hand, suppose that $x \in B_f(p)$, then there exists $n \in \mathbb{Z}^+$ with $f^n(x) \in B_\epsilon(p)$ (since p is attracting), so that $x \in f^{-n}(B_\epsilon(p))$ and $x \in \cup_{n=1}^{\infty} f^{-n}(B_\epsilon(p))$. $\square$

For metric spaces, the idea of “sameness” is given by the notion of “homeomorphism”. Two metric spaces are regarded as “topologically” the same if there is a homeomorphism between them.

Definition 4.3.6 The function $h : X \rightarrow Y$ between metric spaces is a *homeomorphism* if h is one-to-one, onto and both h and h^{-1} are continuous. The spaces X and Y are said to be *homeomorphic*, when there is a homeomorphism between them.

Examples 4.3.7 1. If $f : [0, 1] \rightarrow [0, 1]$, $f(x) = x^2$ then f is a homeomorphism. In fact any strictly increasing continuous function $f : [0, 1] \rightarrow [0, 1]$ with $f(0) = 0$ and $f(1) = 1$ is a homeomorphism. Also any strictly decreasing continuous function $f : [0, 1] \rightarrow [0, 1]$ with $f(0) = 1$ and $f(1) = 0$ is a homeomorphism and it can be shown that any homeomorphism of $[0, 1]$ to itself is of one of the above two forms. The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$ is a homeomorphism. The inverse function is $f^{-1}(x) = x^{1/3}$.

2. The function $f : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$, $f(x) = \arctan x$ is a homeomorphism of the respective metric spaces. The logistic map $L_\mu(x) = \mu x(1 - x)$, $0 < \mu \leq 4$, is not a homeomorphism of $[0, 1]$ as it is not one-to-one (it is only onto when $\mu = 4$).

3. Consider the metric spaces $X = [0, 1]$ and $\Sigma = \{(s_1, s_2, s_3, \dots) : s_i = 0 \text{ or } 1\}$ with the usual metrics. We shall show that these two metric spaces are not homeomorphic.

Exercises 4.3

1. Let $f : \mathbb{R} \rightarrow (-1, 1)$ be defined by:

$$f(x) = \frac{x}{1 + |x|}.$$

- (a) Show that f is a homeomorphism and find the inverse map.
 (b) Extend f to a function $f : [-\infty, \infty] \rightarrow [-1, 1]$ by setting $f(-\infty) = -1$ and $f(\infty) = 1$, and define a metric d on $[-\infty, \infty]$ by

$$d(x, y) = |f(x) - f(y)| \quad \text{for all } x, y \in [-\infty, \infty].$$

Show that d defines a metric on this space whose restriction to $\mathbb{R}$ is different to the usual metric defined on $\mathbb{R}$. $[-\infty, \infty]$ with this metric is called the *extended real line*.

2. Let $f : X \rightarrow X$ be a map defined on the metric space (X, d) and let $\alpha \in \mathbb{R}^+$ be fixed, with the property:

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

Show that f is continuous on X .

3. A map $f : X \rightarrow X$ defined on the metric space (X, d) satisfying

$$d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in X,$$

is called an *isometry*. Show that f is continuous, one-to-one, and hence a homeomorphism onto its range. What are the isometries $f : \mathbb{R} \rightarrow \mathbb{R}$ ($\mathbb{R}$ with the usual metric)?

4. Show that if $f : [a, b] \rightarrow [a, b]$ is a homeomorphism, then either a and b are fixed points or $\{a, b\}$ is a 2-cycle.

5. Let $f : X \rightarrow X$ be a continuous function on a metric space (X, d) . Use the definition of continuity, limits and limit point, in this context to prove:

- (a) If $\lim_{n \rightarrow \infty} x_n = a$, then show that $\lim_{n \rightarrow \infty} f(x_n) = f(a)$.

- (b) If $\lim_{n \rightarrow \infty} f^n(x_0) = p$, then p is a fixed point of f .
- (c) If there is exactly one limit point of the set $O(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$, then it is a fixed point of f .

6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous map with fixed point c and basin of attraction $B_f(c) = (a, b)$ an interval. Show that one of the following must hold:

- (a) a and b are fixed points.
- (b) a or b is fixed and the other is eventually fixed.
- (c) $\{a, b\}$ is a 2-cycle.

7. Let $f : X \rightarrow Y$ be a continuous map of a metric spaces. Prove that if C is a closed set in Y , then $f^{-1}(C)$ is a closed set in X . (Hint: Use the fact that $f^{-1}(Y - U) = X - f^{-1}(U)$, and properties of open sets).

8. Let $f : X \rightarrow X$ be a continuous map of the metric space (X, d) . $x \in X$ is a *wandering point* of f if there is an open set U containing x such that $U \cap f^n(U) = \emptyset$ for all $n \in \mathbb{Z}^+$. A point $x \in X$ is called *non-wandering* if it is not wandering. The set of non-wandering points of f is denoted by $\Omega(f)$. Show the following:

- (a) The set of all wandering points is open. Deduce that $\Omega(f)$ is a closed set which contains all of the periodic points of f .
- (b) $\Omega(f)$ is an invariant set.
- (c) $\Omega(f^n) \subset \Omega(f)$ for all $n \in \mathbb{Z}^+$.
- (d) If f is a homeomorphism, then $f(\Omega(f)) = \Omega(f)$ and $\Omega(f^{-1}) = \Omega(f)$.

9. Let $f : X \rightarrow X$ be a continuous map of the metric space (X, d) . If $x \in X$, then the ω -limit (*omega-limit*) set $\omega(x)$ is defined to be

$$\omega(x) = \bigcap_{n=0}^{\infty} \overline{\{f^k(x) : k > n\}}.$$

It can be shown that if $X = [0, 1]$, then $\omega(x) \subset \Omega(f)$ for all $x \in X$. Show the following:

- (a) $y \in \omega(x)$ if and only if there is an increasing sequence $\{n_k\}$ such that $f^{n_k}(x) \rightarrow y$ as $k \rightarrow \infty$.
- (b) $\omega(x)$ is a closed invariant set ($f(\omega(x)) \subset \omega(x)$).
- (c) If x is a periodic point of f , then $\omega(x) = O(x)$. Also if x is eventually periodic with $y \in O(x)$, then $\omega(x) = O(y)$.
- (e) If $\omega(x)$ consists of a single point, then that point is a fixed point.

10. Let $f : X \rightarrow X$ be a continuous map of the metric space (X, d) . If $x \in X$, it is said to be *recurrent* if $x \in \omega(x)$, i.e., x belongs to its ω -limit set. Recurrent points are a generalization of periodic points. Periodic points return to themselves, whereas recurrent points return closely to themselves infinitely often, becoming closer as the iteration procedure progresses. Show

- (a) A periodic point is recurrent but an eventually periodic point is not recurrent.
- (b) Any recurrent point x belongs to $\Omega(f)$.
- (c) If $f : [0, 1] \rightarrow [0, 1]$ is a homeomorphism, then the only recurrent points are the periodic points.
- (d) Isolated points are recurrent.
- (e) $\omega(x) \subset \overline{O(x)}$ and $\omega(x) = \overline{O(x)}$ if and only if x is recurrent (Need X compact?).

11. Let $f : X \rightarrow X$ be a continuous map of the metric space (X, d) . f is said to be *minimal* if $\overline{O(x)} = X$ for every $x \in X$. Prove that f is minimal if and only if every non-empty closed invariant subset A of X satisfies $A = X$. (Hint: for the $\Leftarrow$ direction, use 8(e) above). We will show later that rotations $R_\alpha : [0, 1) \rightarrow [0, 1)$, $R_\alpha(x) = x + \alpha \pmod{1}$, are minimal when α is irrational.

12. Let $f : X \rightarrow X$ be a map of the metric space (X, d) . f is *transitive* if there exists $x \in X$ with $\overline{O(x)} = X$, i.e., there is a point having a dense orbit. We will show

later that the logistic map L_4 and the tent map T_2 are transitive, and that this is an ingredient of the map being *chaotic*.

(a) Prove that a transitive map is onto.

(b) If f is an isometry which is transitive, prove that f is minimal.

13. Let $f : X \rightarrow X$ be a map of the metric space (X, d) . A non-empty subset $A \subset X$ is said to be *minimal* if the restriction of f to A is a minimal map. It can be shown that any continuous map $f : X \rightarrow X$ of a metric space has at least one minimal set. Prove:

(a) Two minimal sets A and B for f are either disjoint or equal.

(b) If $f : X \rightarrow X$ is an isometry and $\omega(x) \neq \emptyset$, then x is a recurrent point and $\omega(x)$ is a minimal set.

4.4 Diffeomorphisms of $\mathbb{R}$

We study homeomorphisms that are also differentiable functions. If f is a homeomorphism, then the inverse function f^{-1} exists with $y = f(x)$ if and only if $x = f^{-1}(y)$. Denote by I and J open subintervals of $\mathbb{R}$, e.g., $I = (a, b)$ for some $a, b \in \mathbb{R}$, ($a < b$), or $I = (a, \infty)$ etc.

Definition 4.4.1 1. A function $f : I \rightarrow J$ is said to be of class C^1 on I if $f'(x)$ exists and is continuous at all $x \in I$. Such a function is also said to be *smooth*.

2. A homeomorphism $f : I \rightarrow J$ is called a *diffeomorphism* on I if f and f^{-1} are both C^1 -functions on I . If f is a diffeomorphism on $\mathbb{R}$ and $I \subset \mathbb{R}$ is some closed interval, we sometimes refer to f as a diffeomorphism on I , onto the closed interval J where $f(I) = J$.

If $y = f(x)$, then $f^{-1}(f(x)) = x$, so applying the chain rule gives

$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \quad \text{or} \quad (f^{-1})'(y) = \frac{1}{f'(x)}.$$

It follows that for a diffeomorphism we must have $f'(x) \neq 0$ for all $x \in I$. Since $f'(x)$ is a continuous function, if it were allowed to be both positive and negative, then by the Intermediate Value Theorem it must also take the value 0, a contradiction. It

follows that a diffeomorphism has either $f'(x) > 0$ always or $f'(x) < 0$ always. This proves:

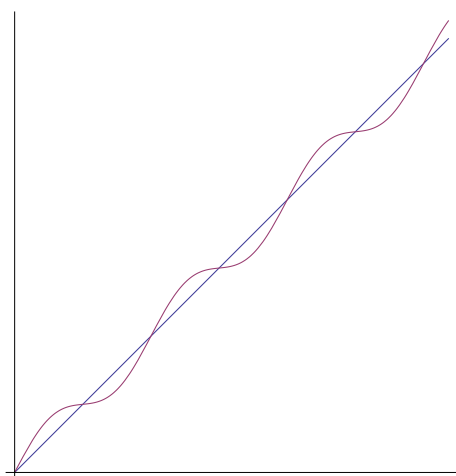
Proposition 4.4.2 *A diffeomorphism $f : I \rightarrow J$ is either:*

1. *Order preserving: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ (f is a strictly increasing function) or*
2. *Order reversing: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ (f is a strictly decreasing function).*

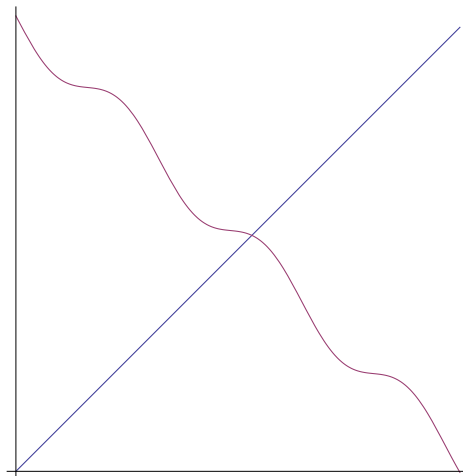
Examples 4.4.3 1. The function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$ is a C^1 function but it is not a diffeomorphism since $f'(0) = 0$, so the inverse function $f^{-1}(x) = x^{1/3}$ is not differentiable at $x = 0$ (vertical tangent).

We can see that $f : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$, $f(x) = \arctan x$ is a diffeomorphism. The inverse function is $f^{-1} : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$, $f^{-1}(x) = \tan x$

2. Order preserving diffeomorphisms can have any number of fixed points, but we see that this is not true in the order reversing case:



Order preserving.



Order reversing.

Theorem 4.4.4 *Let I and J be open intervals and $f : I \rightarrow J$ an order reversing diffeomorphism with $f(I) = J \subseteq I$, then f has a unique fixed point in I .*

Proof. If $I = (a, b)$, then a continuity argument similar to that in Theorem 1.2.9 is enough to ensure the existence of a fixed point. If $I = \mathbb{R}$ let $\alpha = \lim_{x \rightarrow -\infty} f(x)$ and $\beta = \lim_{x \rightarrow \infty} f(x)$ (α and β could be $\pm\infty$). Then $\alpha > \beta$ since f is order reversing.

Let $g(x) = f(x) - x$, then

$$\lim_{x \rightarrow -\infty} g(x) = \infty, \quad \text{and} \quad \lim_{x \rightarrow \infty} g(x) = -\infty,$$

so by IVT there exists $c \in \mathbb{R}$ with $g(c) = 0$, so $f(c) = c$.

The situation for other types of intervals is similar.

Suppose now f is order reversing and has two fixed points, say $f(\alpha) = \alpha$ and $f(\beta) = \beta$ with $\alpha < \beta$, then $f(\alpha) > f(\beta)$ or $\alpha > \beta$ since they are fixed points, a contradiction. $\square$

Examples 4.4.5 Order preserving diffeomorphisms have been seen to have any number of fixed points. We see below that they cannot have points of period greater than 1. On the other hand, order reversing diffeomorphisms can have points of period 2, e.g., consider $f(x) = -x$, but no points of greater period. The bottom line is that the dynamics of one-dimensional diffeomorphisms is not complicated. This is not the case for two-dimensional diffeomorphisms (i.e., diffeomorphisms $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$).

Theorem 4.4.6 *Let $f : I \rightarrow I$ be a diffeomorphism on an open interval I .*

- (i) *If f is order preserving, then f has no periodic points of period greater than 1.*
- (ii) *If f is order reversing, then f has no periodic points of period greater than 2.*

Proof. (i) Let $O(x_0) = \{x_0, x_1, x_2, \dots\}$ be the orbit of x_0 and suppose $x_1 = f(x_0) \neq x_0$.

If $x_1 > x_0$ then $f(x_1) > f(x_0)$ or $x_2 > x_1$. Repeating this argument gives

$$x_0 < x_1 < x_2 < \dots < x_n, \quad \text{so that} \quad f^n(x_0) = x_n \neq x_0.$$

On the other hand, if $x_1 < x_0$, then $f(x_1) < f(x_0)$ so $x_2 < x_1$ and

$$x_0 > x_1 > x_2 > \dots > x_n, \quad \text{so that} \quad f^n(x_0) = x_n \neq x_0.$$

(ii) Notice that if f is an order reversing diffeomorphism, then f^2 is an order preserving diffeomorphism. In fact if $f'(x) < 0$, then

$$(f^2)'(x) = f'(f(x)) \cdot f'(x) > 0,$$

so by (i), f^2 has no n -cycles with $n > 1$, so that f has no $2n$ -cycles with $n > 1$.

If n is odd, then f^n is a diffeomorphism with $(f^n)'(x) < 0$, so that f^n has a unique fixed point which must be the fixed point of f . $\square$

We can improve upon some of the above results in the following way:

Proposition 4.4.7 *Let $f : [a, b] \rightarrow [a, b]$ ($a < b$) be a continuous and one-to-one function.*

- (i) *Either f is strictly increasing or strictly decreasing on $[a, b]$.*
- (ii) *If f is strictly increasing, then every periodic point of f is a fixed point of f .*
- (iii) *If f is strictly decreasing, then f has exactly one fixed point and all other periodic points have period 2.*

Proof. (i) Suppose that $f(a) < f(b)$ (they cannot be equal as f is one-to-one), and let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ and $f(x_1) > f(x_2)$. Suppose that $f(a) < f(x_2) < f(x_1)$, then by the Intermediate Value Theorem there exists $c \in (a, x_1)$ with $f(c) = f(x_2)$, contradicting the one-to-oneness of f , so this is not possible. Similarly if $f(x_2) < f(a) < f(x_1)$ we find $c \in (x_1, x_2)$ with $f(c) = f(a)$. Other possibilities are treated in a similar way.

(ii) Suppose that c is a period point of f having period p . If $c < f(c)$, then since f is strictly increasing, $f(c) < f^2(c)$ and we get an increasing sequence whose limit exists (say L). Then

$$c < L = \lim_{n \rightarrow \infty} f^n(c) = \lim_{n \rightarrow \infty} f^{np}(c) = c,$$

a contradiction. Similarly $f(c) < c$ leads to a contradiction, so $f(c) = c$ and $p = 1$.

(iii) Since f is continuous on a compact interval into itself, f has at least one fixed point. We must have $f(a) > a$ since otherwise $a = f(a) > f(b) \geq a$ which is impossible. Similarly $f(b) < b$.

If c_1 and c_2 are different fixed points of f with $c_1 < c_2$, then $c_1 = f(c_1) > f(c_2) = c_2$, which is impossible, so f has exactly one fixed point. Now notice that since f is order reversing, f^2 is order preserving, so the only periodic points f^2 can have are fixed points, and these are period 2 points of f . $\square$

Remarks 4.4.8 1. It follows that if $f : [a, b] \rightarrow [a, b]$ is a homeomorphism, then it is either strictly increasing or strictly decreasing. If it is increasing, f is order preserving with $f(a) = a$, $f(b) = b$ and the only other periodic points are fixed points. If it is decreasing (order reversing), then it has exactly one fixed point with all other periodic points having period 2. Also, f^2 is orientation preserving and we must have $f(a) = b$, $f(b) = a$. See [46] for a discussion of one-to-oneness.

Exercises 4.4

1. Show that $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$ is not a diffeomorphism of $\mathbb{R}$. Give an example of a diffeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$.

2*. Show that $f(x) = 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n!$ is a diffeomorphism of $\mathbb{R}$ when n is odd.

4.5 Countability, Sets of Measure Zero and the Cantor Set

Given an infinite set A , there are different ways of thinking of A as being large. For example, we might say that $\mathbb{Q} \subset \mathbb{R}$ is a large set since it is dense in $\mathbb{R}$. However, in some sense it is a small set, what we call a *countable set* and also a set of *measure zero*. Sets of measure zero are sets that are small in terms of “length”. Many of the ideas in this section are due to George Cantor, a German mathematician of the 19th century. Before we examine the notion of countability, we look at sets of measure zero.

Definition 4.5.1 Let I be a bounded subinterval of $\mathbb{R}$, having end-points a, b ($a \leq b$). The *length* of I is then $|I| = b - a$. If I is an unbounded interval, we set $|I| = \infty$.

Definition 4.5.2 We say that a set $A \subseteq \mathbb{R}$ is a *set of measure zero* if we can cover A by bounded open intervals indexed by the set $\mathbb{Z}^+$, so that the total length of the intervals can be chosen to be arbitrarily small. More precisely, given any $\epsilon > 0$ there is a collection of open intervals $\{I_n : n \in \mathbb{Z}^+\}$ with

$$A \subseteq \bigcup_{n \in \mathbb{Z}^+} I_n \quad \text{and} \quad \sum_{n \in \mathbb{Z}^+} |I_n| \leq \epsilon.$$

A collection of open sets O_λ , ($\lambda \in \Lambda$), whose union contains a set A (in a metric space) is called an *open cover* of the set A .

The requirement that the intervals be open is not a serious one, they may be closed or half open. For example, if the intervals I_n are closed, replace I_n by J_n with $I_n \subset J_n$ and $|J_n| = |I_n| + \epsilon/2^n$, then

$$\sum_{n \in \mathbb{Z}^+} |J_n| = \sum_{n \in \mathbb{Z}^+} |I_n| + \sum_{n \in \mathbb{Z}^+} \epsilon/2^n \leq 2\epsilon.$$

If a property P holds everywhere except for a set of measure zero, we sometimes say that it holds *almost everywhere* (abbreviated a.e.), so for example, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases},$$

then we say $f(x) = 0$ a.e.

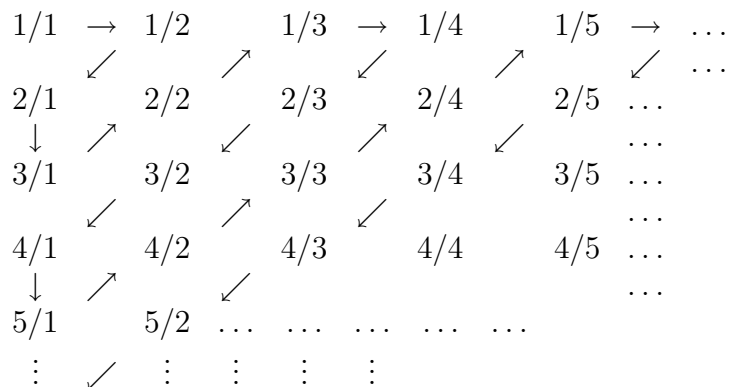
Example 4.5.3 1. A finite subset $A = \{a_1, a_2, \dots, a_n\}$ of $\mathbb{R}$ is a set of measure zero because if $\epsilon > 0$, we can find open intervals say $I_i = (a_i - \epsilon/2n, a_i + \epsilon/2n)$, $i = 1, 2, \dots, n$, with $A \subset \cup_{i=1}^n I_i$ and the total length of the intervals being ϵ .

2. The set $\mathbb{Q}$ of rationals, is a set of measure zero. To prove this we make a digression on countability. Countably infinite sets (said to be *denumerable*) are in a sense the smallest type of infinite set.

Definition 4.5.4 Any set A that is finite or can be put into a one-to-one correspondence with the set $\mathbb{Z}^+$ is said to be *countable*. This is equivalent to saying that the members of A may be listed as a sequence.

Examples 4.5.5 1. If $A = \{1, 4, 9, 16, \dots, n^2, \dots\}$, then A is countable since we can define $f : \mathbb{Z}^+ \rightarrow A$ by $f(n) = n^2$, a one-to-one and onto map. Any subset of a countable set can be seen to be countable.

2. The set $\mathbb{Q}$ of rational numbers is a countable set. We list the positive rationals as a sequence (with some repetitions) by following the arrows below. The general case can be demonstrated with some minor modifications:



3. The set $\mathbb{R}$ is not countable (it is said to be *uncountable*). In fact we show (by contradiction) that the interval $[0, 1]$ is uncountable:

Suppose that $[0, 1]$ is a countable set, then we can list its members as a sequence $x_1, x_2, x_3, \dots, x_n, \dots$. Each member of this sequence has a binary expansion, say

$$\begin{array}{rcl} x_1 & = & \cdot a_{11} a_{12} a_{13} a_{14} \dots \\ x_2 & = & \cdot a_{21} a_{22} a_{23} a_{24} \dots \\ x_3 & = & \cdot a_{31} a_{32} a_{33} a_{34} \dots \\ \vdots & & \vdots \\ x_n & = & \cdot a_{n1} a_{n2} a_{n3} a_{n4} \dots \\ \vdots & & \vdots \end{array}$$

Let $y = .b_1 b_2 b_3 b_4 \dots b_n \dots$, where $b_i = \begin{cases} 0 & \text{if } a_{ii} = 1 \\ 1 & \text{if } a_{ii} = 0 \end{cases}$, then y differs from x_1 in the first binary digit, it differs from x_2 in the second binary digit etc., differing from x_n in the n th binary digit, so y does not appear anywhere in the sequence, contradicting the fact that we have listed all members of $[0, 1]$ as a sequence.

Remarks 4.5.6 1. Each $x \in (0, 1)$ has a unique *non-terminating* binary expansion, for example:

$$\frac{1}{2} = .10000\dots, \quad \text{and} \quad \frac{1}{2} = .0111\dots,$$

so $1/2$ has two different binary representations, but only the second is non-terminating. In the proof of the uncountability of $[0, 1]$ we should assume the x_i have a non-terminating expansion. This gives $y = .b_1 b_2 b_3 b_4 \dots b_n \dots$, but we don't know if it is terminating or not. To avoid this difficulty we should start by listing all possible binary expansions (both terminating and non-terminating), so some numbers are listed twice, but there are only countably many having terminating expansions (these are the binary rationals such as $1/2$, $3/4 = .1100\dots$ etc.) Now proceed as previously to see that y is not one of the enumerated elements.

2. It is easy to see that there is a one-to-one correspondence between the interval $(0, 1)$ and $\mathbb{R}$ (we say these sets have the same *cardinality*). For example, the map $F : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$, $F(x) = \arctan(x)$ is both one-to-one and onto. If we take a linear (affine) map between $(-\pi/2, \pi/2)$ and $(0, 1)$, the result is clear.

We can now prove that the rationals (and in fact any countable set) is a set of measure zero, i.e., can be covered by a collection of open intervals whose total length can be made as small as we like. It can be shown that an interval such as $[0, 1]$ is not a set of measure zero and also that the irrational numbers in $[0, 1]$ do not constitute a set of measure zero.

Proposition 4.5.7 *The rational numbers form a set of measure zero.*

Proof. We can enumerate $\mathbb{Q}$ as a sequence $r_1, r_2, r_3, \dots, r_n, \dots$, say. Let $\epsilon > 0$ and for each $n \in \mathbb{Z}^+$ define an interval I_n by

$$I_n = (r_n - \epsilon/2^{n+1}, r_n + \epsilon/2^{n+1}).$$

Clearly $r_n \in I_n$, $n = 1, 2, \dots$, so that $\mathbb{Q} \subset \cup_{i=1}^{\infty} I_n$. The total length of these intervals is

$$\sum_{i=1}^{\infty} |I_n| = \sum_{i=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon,$$

which can be made as small as we please. The set $E = \cup_{n=1}^{\infty} I_n$ is another example of a set in $\mathbb{R}$ which is both open and dense, containing $\mathbb{Q}$ and not equal to $\mathbb{R}$. $\square$

We now show that an interval I , with end-points a and b , $a < b$, cannot be a set of measure zero. The following result is due to Borel (see Oxtoby [47]). We first need a lemma which is of interest in its own right, and is related to a result known as the *Heine-Borel Theorem*:

Lemma 4.5.8 *Let $\{O_\lambda\}_{\lambda \in \Lambda}$ be a collection of open sets which cover the interval $[a, b]$, $a \leq b$ (i.e., $[a, b] \subset \cup_{\lambda \in \Lambda} O_\lambda$). Then there is a finite subcollection of these sets that cover $[a, b]$.*

Proof. Let

$$\mathcal{S} = \{x \in [a, b] : \text{there is a finite subcollection of intervals from } O_\lambda, \lambda \in \Lambda \text{ that cover } [a, x]\}.$$

Clearly $a \in \mathcal{S}$ and the set $\mathcal{S}$ is bounded above by b , so $\alpha = \sup(\mathcal{S})$ exists and $\alpha \leq b$. Suppose that $\alpha < b$, then α belongs to some open set say O_μ from our collection. Since this set is open, there is an open interval $(\alpha - \epsilon, \alpha + \epsilon) \subset O_\mu$. If $\beta \in (\alpha, \alpha + \epsilon)$, then it is clear that we can cover the interval $[a, \beta]$ by a finite subcollection of the open sets since we can always include O_μ . This contradicts the fact that α is the least upper bound of the set $\mathcal{S}$, so we must have $\alpha = b$. $\square$

Proposition 4.5.9 *If a finite or infinite sequence of interval I_n , $n = 1, 2, \dots$, covers an interval I with end points $a, b \in \mathbb{R}$, $a < b$, then $\sum_n |I_n| \geq |I|$.*

Proof We give the proof for the case where the intervals I_n are open, and $I = [a, b]$ is a closed interval. A slight modification will give the general result.

Denote by (a_1, b_1) the first interval that contains the point a . If $b_1 < b$, let (a_2, b_2) be the first interval in the sequence I_n that contains the point b_1 . Continue in this way so that if $b_{n-1} < b$, (a_n, b_n) is the first interval that contains the point b_{n-1} .

This procedure must terminate with some $b_N > b$, for if not we would have an increasing sequence bounded above by b , so must converge to a limit $\alpha \leq b$, where α belongs to I_k for some k . All but a finite number of the intervals (a_n, b_n) would have to precede I_k in the given sequence, namely, all those for which $b_{n-1} \in I_k$. This is impossible since no two of these intervals are equal.

It follows that

$$b - a < b_N - a_1 = \sum_{i=2}^N (b_i - b_{i-1}) + b_1 - a_1 \leq \sum_{i=1}^N (b_i - a_i),$$

and the result follows. $\square$

We deduce that the interval $[a, b]$, $a < b$, is not a set of measure zero. This also gives an alternative proof of the fact that an interval is an uncountable set:

Corollary 4.5.10 *An interval I with end-points $a, b \in \mathbb{R}$, ($a < b$), is not of measure zero, and hence cannot be a countable set.*

Example 4.5.11 The Cantor Set.

The Cantor set C is defined as follows: Set $S_0 = [0, 1]$, the unit interval. We remove the open interval $(1/3, 2/3)$ from S_0 to give $S_1 = [0, 1/3] \cup [2/3, 1]$. We continue removing *open middle thirds* to give

$$S_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1],$$

and continue in this way removing open middle thirds so that $S_n = [0, 1/3^n] \cup \dots$. Denoting the total length of the of the subintervals making up S_n by $|S_n|$ (so $|S_0| = 1$), we have:

S_1 consists of 2 intervals of total length $|S_1| = 2/3$,

S_2 consists of 2^2 intervals of total length $|S_2| = 2^2/3^2$,

S_3 consists of 2^3 intervals of total length $|S_3| = 2^3/3^3$,

and generally

S_n consists of 2^n intervals of total length $|S_n| = 2^n/3^n$.





The first three steps in the construction of the Cantor Set.

Definition 4.5.12 The Cantor set C is defined by

$$C = \bigcap_{n=1}^{\infty} S_n.$$

Our aim now is to show that the Cantor set is a large set in the sense that it is not countable, but small in the sense that it is a set of measure zero. We also investigate the properties of C , and show a connection between C and the ternary expansion of certain numbers in $[0, 1]$. The Cantor set is an example of a *fractal* - it has the property of *self-similarity*. For example $C \cap [0, 1/3]$ looks exactly like the Cantor set, but on a smaller scale. $C \cap [0, 1/9]$ is a replica of C but on a smaller scale still. We can continue like this indefinitely to see C on smaller and smaller scales.

Proposition 4.5.13 *The Cantor set C is a closed non-empty subset of $[0, 1]$, having measure zero.*

Proof. C is non-empty because it contains all of the end points of each of the intervals constituting the set S_n , $n \in \mathbb{Z}^+$ (for example, $1/3 \in S_n$ for every $n \in \mathbb{Z}^+$). $C \subset S_n$ for $n = 1, 2, \dots$ where $|S_n| = (2/3)^n \rightarrow 0$ as $n \rightarrow \infty$, so that C may be covered by a collection of intervals whose total length can be made arbitrarily small. C is a closed set because each of the sets S_n , $n \in \mathbb{Z}^+$ is closed, and C is the intersection of closed sets. $\square$

Exercises 4.5

1. *Cantor's Ternary Function* is defined as follows: Let $x \in [0, 1]$ have ternary expansion $x = .x_1 x_2 x_3 \dots$ ($x_i \in \{0, 1, 2\}$, $i = 1, 2, 3, \dots$). Set $N = \infty$ if $x_n \neq 1$ for

all $n \in \mathbb{Z}^+$, otherwise set $N = \min\{n \in \mathbb{Z}^+ : x_n = 1\}$. Let $y_n = x_n/2$ for $n < N$ and $y_N = 1$.

(a) Show that $\sum_{n=1}^N y_n/2^n$ is independent of the ternary expansion of x , if x has two such expansions.

(b) Show that the function $\kappa : [0, 1] \rightarrow [0, 1]$ defined by

$$\kappa(x) = \sum_{n=1}^N \frac{y_n}{2^n}$$

is a continuous, onto, and monotone increasing ($\kappa(x) \leq \kappa(y)$ for all $x < y$ in $[0, 1]$).

(c) Show that κ is constant on each interval contained in the complement of the Cantor set (for example, $\kappa(x) = 1/2$ for all $x \in (1/3, 2/3)$).

2. Prove the following about sets of *measure zero*:

- (a) A subset of a set of measure zero has measure zero.
- (b) Any countable set has measure zero.
- (c) The countable union of sets of measure zero has measure zero.

3. Prove that the set $\Sigma = \{(s_1, s_2, s_3, \dots) : s_i = 0 \text{ or } s_i = 1\}$ of all sequences of 0's and 1's is uncountable.

4. Determine whether the set of numbers in $(0, 1)$ whose decimal expansion contains no 1's or 5's is:

- (i) closed, open or neither,
- (ii) totally disconnected.

5. Lemma 4.5.8 shows that if we cover an interval $[a, b]$ with an arbitrary collection of open sets, there is a finite subcollection of these sets that also cover $[a, b]$. A set $A \subseteq \mathbb{R}$ having this property (every open cover has a finite subcover) is said to be

compact, and therefore the interval $[a, b]$ is a compact set. Clearly every finite subset of $\mathbb{R}$ is compact.

(a) Show that the sets (i) $A = [0, \infty)$, (ii) $B = \{1, 1/2, 1/3, \dots\}$ are not compact by exhibiting an open cover that does not have a finite subcover. On the other hand, show that $B \cup \{0\}$ is compact.

(b) Show that any closed bounded subset K of $\mathbb{R}$ is compact (Hint: Use the fact that K is a subset of a closed interval of the form $[a, b]$, and extend an open cover of K to one for $[a, b]$ and then use the compactness of $[a, b]$).

6. Show that a set of measure zero cannot contain an open interval (we say that a subset $A \subset X$ of a metric space X has *empty interior* if it does not contain any open balls).

4.6 Ternary Expansions and the Cantor Set.

Each $x \in [0, 1]$ has a *ternary expansion*, i.e., can be written as

$$x = \cdot a_1 a_2 a_3 a_4 \dots = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \dots, \quad \text{where } a_i \in \{0, 1, 2\}.$$

Note that if $a_1 = 0$ then $x \in [0, 1/3]$ since $x = \cdot 0 a_2 a_3 \dots \leq \cdot 0 2 2 2 \dots = 1/3$. Similarly, if $a_1 = 1$, then $x \in [1/3, 2/3]$, and if $a_1 = 2$, then $x \in [2/3, 1]$. Conversely, if $x \in [0, 1/3]$, the x has a ternary expansion with $a_1 = 0$, and similarly for the other two intervals.

In addition, just as in the binary expansion situation, every $x \in (0, 1)$ has a unique *non-terminating* ternary expansion. For example

$$\frac{1}{3} = \cdot 1000 \dots = \cdot 0222 \dots, \quad \text{and} \quad \frac{2}{3} = \cdot 2000 \dots = \cdot 1222 \dots$$

We shall show that $x \in C$ if and only if x has a ternary expansion (possibly terminating) consisting only of 0's and 2's. This means, for example, that $1/3 \in C$, $1 = \cdot 2222 \dots \in C$, $2/3 \in C$ and $3/4 = \cdot 20202 \dots \in C$.

Theorem 4.6.1 $x \in C$ if and only if x has a ternary expansion

$$x = \cdot a_1 a_2 a_3 \dots, \quad \text{where } a_i = 0 \text{ or } 2.$$

Proof. First suppose that $x \in C$ with a ternary expansion

$$x = \cdot a_1 a_2 a_3 \dots,$$

then $x \in S_n$ for each $n \in \mathbb{Z}^+$. In particular $x \in S_1$, so either $x \in [0, 1/3]$ and we may take $a_1 = 0$, or $x \in [2/3, 1]$ and $a_1 = 2$. Also $x \in S_2$, so either $x \in [0, 1/9]$ and $a_2 = 0$, or $x \in [2/9, 1/3]$ and $a_2 = 2$, or $x \in [2/3, 7/9]$ and $a_2 = 0$, or $x \in [8/9, 1]$ and $a_2 = 2$. Continuing in this way we see that we may take $a_n = 0$ or $a_n = 2$ for each $n \in \mathbb{Z}^+$.

Conversely, suppose that we can choose the expansion so that $a_n = 0$ or $a_n = 2$ for each $n \in \mathbb{Z}^+$, then since this holds for $n = 1$ we must have $x \in [0, 1/3]$ or $x \in [2/3, 1]$ so that $x \in S_1$. Similarly $a_2 = 0$ or 2 implies that $x \in S_2$ and continuing in this way, $x \in S_n$ for $n = 1, 2, \dots$, we deduce that $x \in \cap_{n=1}^{\infty} S_n = C$. $\square$

We can now prove that C is not a countable set (in particular, it can be put into a one-to-one correspondence with $[0, 1]$).

Theorem 4.6.2 *The Cantor set C is uncountable.*

Proof. We define a one-to-one function $f : [0, 1] \rightarrow C$ as follows:

Let $x \in [0, 1]$ have a binary representation (non-terminating)

$$x = \cdot a_1 a_2 a_3 a_4 \dots \quad \text{where } a_i \in \{0, 1\}.$$

Define $f(x)$ by

$$f(x) = \cdot b_1 b_2 b_3 b_4 \dots \quad \text{where } b_i = \begin{cases} 0 & \text{if } a_i = 0, \\ 2 & \text{if } a_i = 1. \end{cases}$$

Then $f(x)$ is one-to-one (it is not be onto because certain numbers such as $2/3 = \cdot 1222 \dots \in C$ are not in the range of f). It follows that there is a subset of C that is uncountable, so C itself is uncountable. $\square$

Finally we show that the Cantor set is *totally disconnected* and *perfect*. In this context, to say that C is totally disconnected means that it contains no intervals, and the fact that it is perfect means that every point of C is a limit point of C . More formally:

Definition 4.6.3 A subset $A \subset \mathbb{R}$ is said to be *totally disconnected* (or has *empty interior*), if A contains no non-empty open intervals. For example, discrete sets of points are totally disconnected, $\mathbb{Q}$ the set of rationals in $\mathbb{R}$ is totally disconnected.

Definition 4.6.4 A subset $A \subset \mathbb{R}$ is said to be *perfect* if every point of A is a limit point of A .

Theorem 4.6.5 *The Cantor set C is totally disconnected and perfect.*

Proof. If U is a non-empty open interval contained in C , then U is contained in S_n for each $n \in \mathbb{Z}^+$. But $|S_n| = (2/3)^n \rightarrow 0$ as $n \rightarrow \infty$, so this is impossible as every non-empty open interval has positive length.

To see that C is perfect, let $x \in C$ have ternary expansion

$$x = \cdot a_1 a_2 \dots a_n \dots \quad \text{where } a_i \in \{0, 2\}.$$

Set

$$x_n = \cdot a_1 a_2 \dots a_n * * * \dots,$$

chosen so that $x_n \in C$ and agrees with x in the first n -places and $x_n \neq x$ and $x_n \neq x_m$ for all $n, m \in \mathbb{Z}^+$, $n \neq m$. Then

$$|x - x_n| \leq \sum_{k=n+1}^{\infty} \frac{2}{3^k} = \frac{1}{3^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

It follows that x is a limit point of C . □

There are other “Cantor Sets” besides C (see Exercises 4.6). Any subset of $\mathbb{R}$ that is closed, bounded, perfect and totally disconnected is said to be a *Cantor Set*. We shall see that these arise quite naturally in dynamical systems theory.

Exercises 4.6

1. (a) Let $\Sigma = \{(a_1, a_2, a_3, \dots) : a_i = 0 \text{ or } 1\}$, the sequence space of zeros and ones with the metric defined in Section 4.1. Let C be the Cantor set and define a map $f : \Sigma \rightarrow C$ by

$$f(a_1, a_2, a_3, \dots) = \cdot b_1 b_2 b_3 \dots, \quad \text{where } b_i = 0 \text{ if } a_i = 0 \text{ and } b_i = 2 \text{ if } a_i = 1,$$

giving the ternary expansion of a real number in $[0, 1]$. Show that this defines a homeomorphism between Σ and the Cantor set.

(b) It can be shown that $[0, 1]$ and C are not homeomorphic (for example, C is totally disconnected whilst $[0, 1]$ is not). What goes wrong if we try to define a homeomorphism between $[0, 1]$ and C by mapping the binary expansion of $x \in [0, 1]$ to the ternary expansion?

2. We have seen that the Cantor set is an uncountable set having measure zero. Here we generalize the notion of Cantor set to give an example of a set having empty interior which does not have measure zero:

Fix $\alpha \in (0, 1)$ and for each $n \in \mathbb{Z}^+$, let $a_n = 2^{n-1}\alpha/3^n$. Set $S_0(\alpha) = [0, 1]$, and from its center, remove an open interval of length a_1 and denote by $S_1(\alpha)$ the resulting set, a union of two disjoint closed intervals. From the center of each of these two intervals, remove an open interval of length $a_2/2$ to obtain a set $S_2(\alpha)$ which is the union of 2^2 disjoint closed intervals of equal length. Continue in this way as in the construction of the Cantor set, so that at the n th stage we have a set $S_n(\alpha)$ obtained by removing 2^{n-1} disjoint closed intervals in $S_{n-1}(\alpha)$ an open interval of length $a_n/2^{n-1}$. Set

$$C(\alpha) = \bigcap_{n=1}^{\infty} S_n(\alpha).$$

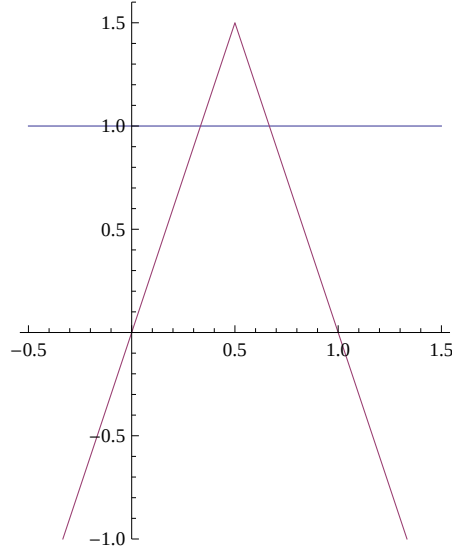
$C(\alpha)$ is a *generalized Cantor set*. It is a subset of $[0, 1]$ which can be shown to be uncountable and perfect. When $\alpha = 1$ we obtain the usual Cantor set. Show:

- (a) $C(\alpha)$ is a closed non-empty set.
- (b) $C(\alpha)$ has empty interior. (Hint: Suppose that $C(\alpha)$ contains an open interval I and show this is impossible since the lengths of the intervals in $C_n(\alpha)$ can be made arbitrarily small by making n large enough).
- (c) $C(\alpha)$ is not a set of measure zero. (Hint: Find the total lengths of the intervals removed in the construction of $C(\alpha)$).

4.7 The Tent Map for $\mu = 3$.

Now consider the tent family T_μ for $\mu = 3$, where we think of T_3 as a map defined on all of $\mathbb{R}$ by

$$T_3(x) = \begin{cases} 3x & \text{if } x \leq 1/2, \\ 3(1-x) & \text{if } x > 1/2. \end{cases}$$



The tent map T_μ with $\mu = 3$.

Note that $\{3/13, 9/13, 12/13\}$ and $\{3/28, 9/28, 27/28\}$ are both 3-cycles for T_3 and that if $x > 1$, then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$. We shall show that the orbit of $x \in [0, 1]$ is bounded if and only if $x \in C$, the Cantor set. In particular, if $x \notin C$, then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$. We also see that C is an invariant set under T_3 ($T_3(C) = C$), so we can consider T_3 as a map $T_3 : C \rightarrow C$, and this is where the interesting dynamics of T_3 takes place.

Proposition 4.7.1 *If $\Lambda = \{x \in [0, 1] : T_3^n(x) \in [0, 1] \forall n \in \mathbb{Z}^+\}$, then $\Lambda = C$, the Cantor set. In addition, $T(C) \subseteq C$.*

Proof.

Claim 1. If $x \in (1/3, 2/3)$ then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

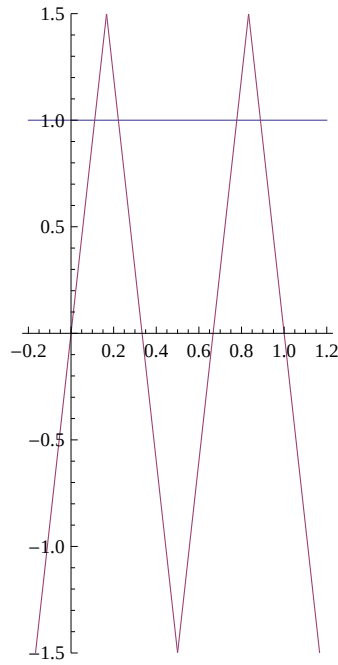
This is because T_3 is increasing on $(1/3, 1/2]$ and decreasing on $[1/2, 2/3)$ so that if $1/3 < x \leq 1/2$, then $T_3(1/3) < T_3(x) \leq T_3(1/2)$ so $1 < T_3(x) \leq 3/2$.

Similarly if $1/2 \leq x < 2/3$ then $1 < T_3(x) \leq 3/2$, so $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

Claim 2. If $x \in (1/9, 2/9) \cup (7/9, 8/9)$, then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

We use the fact that

$$T_3^2(x) = \begin{cases} 9x & \text{if } x \leq 1/6, \\ 3 - 9x & \text{if } 1/6 \leq x < 1/2, \\ 9x - 6 & \text{if } 1/2 \leq x < 5/6, \\ 9 - 9x & \text{if } x \geq 5/6, \end{cases}$$

The tent map T_3^2 .

T_3^2 is strictly increasing on $(1/9, 1/6]$ and strictly decreasing on $[1/6, 2/9)$, so that if $1/9 < x \leq 1/6$, then $T_3^2(1/9) < T_3^2(x) \leq T_3^2(1/6)$ and $1 < T_3^2(x) \leq 3/2$ and similarly if $1/6 < x \leq 2/9$ then $1 < T_3^2(x) \leq 3/2$. A similar argument also applies when $x \in (7/9, 8/9)$ to give $T_3^2(x) > 1$ in each case, so that $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

Claim 3 If $x \notin C$, then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

We have seen that if $x \in (1/3, 2/3)$ then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$. Suppose instead $x \in [0, 1/3] \cup [2/3, 1]$ and has the ternary expansion

$$x = \cdot a_1 a_2 a_3 a_4 \dots \quad \text{where } a_i \in \{0, 1, 2\},$$

then

$$T_3(x) = \begin{cases} \cdot a_2 a_3 a_4 \dots & \text{if } 0 \leq x \leq 1/3 \\ \cdot b_2 b_3 b_4 \dots & \text{if } 2/3 \leq x \leq 1, \end{cases}$$

where $b_i = \begin{cases} 0 & \text{if } a_i = 2 \\ 1 & \text{if } a_i = 1 \\ 2 & \text{if } a_i = 0 \end{cases}$. Consequently, if x has a 1 in its ternary expansion, there exists $k \in \mathbb{Z}^+$ with

$$T_3^k(x) = \cdot 1 a_{k+1} \dots, \quad \text{and so } T_3^k(x) \in [1/3, 2/3).$$

We can only have $T_3^k(x) = 1/3$ when the ternary expansion of x consists of a sequence of 0's and 2's (k terms), followed by a 1, and then followed by an infinite string of 0's (in this case $x \in C$ because the terms $1000\dots$ can be written as $0222\dots$). Thus $T_3^{k+1}(x) > 1$ and the orbit of x will go to $-\infty$. Consequently, we deduce that if $x \notin C$ then $T_3^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$.

Claim 4. If $x \in C$, then $T_3(x) \in C$.

If each $a_i = 0$ or 2 , then each b_i (in Claim 3) is 0 or 2 , so the result follows. $\square$

In conclusion, we see that we may consider T_3 as a map $T_3 : C \rightarrow C$. The Cantor set C is the set on which the interesting dynamics takes place. We can see that T_3 is an onto map, but it is clearly not one-to-one. A similar analysis can be made for T_μ when $\mu > 2$ to deduce that there is some set C_μ , a type of Cantor set, on which the interesting dynamics takes place. Similar (but more complicated) reasoning also shows that the logistic map $L_\mu = \mu x(1 - x)$, $\mu > 4$, $x \in \mathbb{R}$ has the same type of behavior.

4.8 Another Cantor Set

We look at a Cantor set arising from the logistic map $L_\mu(x) = \mu x(1 - x)$ for $\mu > 4$. We think of L_μ as a function defined on all of $\mathbb{R}$, but are mainly interested in its restriction to $[0, 1]$. Now $L_\mu(1/2) = \mu/4 > 1$, and $x = 1/2$ is a critical point for L_μ . From the graph of L_μ we see that there are two point a_0 and $a_1 = 1 - a_0$ where $L_\mu(a_0) = 1 = L_\mu(a_1)$, and in addition, $L_\mu(x) > 1$ for all $x \in (a_0, a_1)$. Just as for the tent map T_3 , we see that if $x \in (a_0, a_1)$, then $L_\mu^n(x) \rightarrow -\infty$ as $n \rightarrow \infty$. Set $I_0 = [0, a_0]$ and $I_2 = [a_1, 1]$ and $\Lambda_1 = I_0 \cup I_2$, a disjoint union of two closed interval, so is a closed set:

$$\Lambda_1 = \{x \in [0, 1] : L_\mu(x) \in [0, 1]\}.$$

We continue in this way as we did with the tent map T_3 to find a set Λ_2 , a disjoint union of 4 closed intervals:

$$\Lambda_2 = \{x \in [0, 1] : L_\mu^2(x) \in [0, 1]\}.$$

And at the n th stage

$$\Lambda_n = \{x \in [0, 1] : L_\mu^n(x) \in [0, 1]\}$$

is the disjoint union of 2^n closed intervals. We set $\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n$, the intersection of closed sets, so is a closed set contained in $[0, 1]$ (the fact that the set Λ_n is closed also follows from the fact that it is the inverse image of a closed set - see the following paragraph).

Recall that if $f : X \rightarrow X$ is a function on a metric space, the inverse image of a set $U \subseteq X$ is the set $f^{-1}(U) = \{x \in X : f(x) \in U\}$. We see that

$$\Lambda_1 = L_\mu^{-1}([0, 1]), \Lambda_2 = L_\mu^{-2}([0, 1]), \dots, \Lambda_n = L_\mu^{-n}([0, 1]),$$

$$\Lambda = \cap_{n=1}^{\infty} \Lambda_n = \cap_{n=1}^{\infty} L_\mu^{-n}[0, 1],$$

so that $x \in \Lambda$ if and only if $L_\mu^n(x) \in [0, 1]$ for all $n \in \mathbb{Z}^+$. Also, Λ is a non-empty set because it must contain the period points of L_μ . It can be shown to be a perfect and totally disconnected set of measure zero, so it is a Cantor set. As before we can consider L_μ as a map $L_\mu : \Lambda \rightarrow \Lambda$, as this is where the interesting dynamics takes place.

The above analysis can be made easier in the case where $\mu > 2 + \sqrt{5}$. In this situation, if we find the points a_0 and a_1 by setting $L_\mu(x) = 1$, we get the equation $\mu x^2 - \mu x + 1 = 0$. Solving gives

$$a_i = \frac{\mu \pm \sqrt{\mu^2 - 4\mu}}{2\mu}, \quad i = 0 \text{ or } 1.$$

For $x \in \Lambda_1$, $|L'_\mu(x)|$ is a minimum when $x = a_0$, or $x = a_1$, and in this case

$$L'_\mu(a_0) = \mu - 2\mu a_0 = \sqrt{\mu^2 - 4\mu}.$$

Thus for $\mu > 0$ and $x \in \Lambda_1$,

$$L'_\mu(x) \geq L'_\mu(a_0) = \sqrt{\mu^2 - 4\mu} > 1,$$

when $\mu^2 - 4\mu - 1 > 0$, for which we require $\mu > 2 + \sqrt{5}$. This shows that L_μ cannot have any attracting periodic points if $\mu > 2 + \sqrt{5}$. It can now be shown that the individual intervals making up each set Λ_n has length less than $1/r^n$, for some $r > 1$ and this is used to show that Λ is a Cantor set. This is also true for any $\mu > 4$, but the analysis is more difficult. (see [20] for more details)

Chapter 5. Devaney's Definition of Chaos

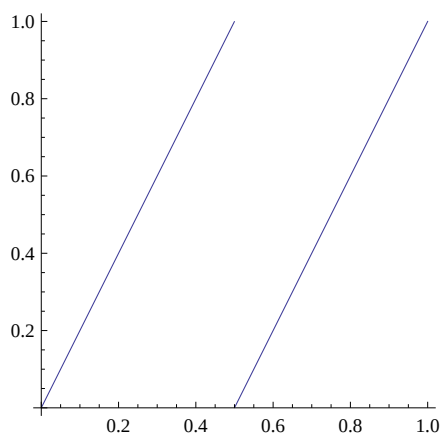
In this chapter we will give Devaney's definition of chaos for one-dimensional maps and also for more general maps defined on metric spaces. One-dimensional maps are functions $f : I \rightarrow \mathbb{R}$ for some interval $I \subseteq \mathbb{R}$. It turns out that maps whose periodic points form a dense set often have highly chaotic properties.

5.1 The Doubling Map and the Angle Doubling Map

Example 5.1.1 The Doubling Map.

The doubling map $D : [0, 1] \rightarrow [0, 1]$ is defined by

$$D(x) = 2x \pmod{1} = \begin{cases} 2x ; & 0 \leq x < 1/2, \\ 2x - 1 ; & 1/2 \leq x \leq 1. \end{cases}$$



The doubling map D .

It is useful to describe D in terms of the binary expansion of a real number in $[0, 1]$. Let $x \in [0, 1]$ with binary expansion

$$x = .a_1a_2a_3 \dots \quad \text{where } a_i = 0 \text{ or } 1.$$

In other words,

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

Suppose that $a_1 = 0$ in the binary expansion of x , then

$$D(x) = 2x = \frac{a_2}{2} + \frac{a_3}{2^2} + \dots = .a_2a_3 \dots$$

On the other hand, if $a_1 = 1$, then

$$D(x) = 2x - 1 = (a_1 + \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots) - 1 = \frac{a_2}{2} + \frac{a_3}{2^2} + \cdots = \cdot a_2 a_3 \dots$$

We see that in general

$$D(\cdot a_1 a_2 a_3 \dots) = \cdot a_2 a_3 \dots \quad \text{and} \quad D^n(\cdot a_1 a_2 a_3 \dots) = \cdot a_{n+1} a_{n+2} a_{n+3} \dots$$

Consequently, if $x = \cdot a_1 a_2 \dots a_n a_1 a_2 \dots a_n a_1 \dots$ has an expansion which repeats every n places, then $D^n(x) = x$, so that x is periodic of period n .

For example $D^2(\cdot 010101 \dots) = \cdot 010101 \dots$, so is a point of period 2. We use this to show that the set of periodic points of D are dense in $[0, 1]$. This can also be used to count the number of periodic points of period n . Notice that since $D'(x) > 1$ everywhere it is defined, all of the periodic orbits of D are unstable.

Proposition 5.1.2 *The periodic points of the doubling map are dense in $[0, 1]$.*

Proof. Let $\epsilon > 0$ and choose N so large that $1/2^N < \epsilon$. If $x \in [0, 1]$ it suffices to show that there is a periodic point y for D that is within ϵ of x . Suppose that the binary expansion of x is

$$x = \cdot a_1 a_2 a_3 \dots = \sum_{i=1}^{\infty} \frac{a_i}{2^i},$$

then we set

$$y = \cdot a_1 a_2 \dots a_N a_1 a_2 \dots a_N a_1 \dots,$$

a point of period N . Then

$$|x - y| = \left| \sum_{j=N+1}^{\infty} \frac{b_j}{2^j} \right| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^N} < \epsilon,$$

(where $b_i = 0, 1$ or -1).

Example 5.1.2 The Angle Doubling Map.

As before, we denote by $\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$, the set of all complex numbers. If $z = a + ib \in \mathbb{C}$, then its *absolute value* (or *modulus*) is given by $|z| = \sqrt{a^2 + b^2}$. The conjugate of z is $\bar{z} = a - ib$ and we can check that $z\bar{z} = |z|^2$. We can represent $\mathbb{C}$ using points in the (*complex*) *plane* $\{(a, b) : a, b \in \mathbb{R}\}$. The *unit circle* S^1 in the complex plane is the set

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Points in S^1 may be represented as:

$$z = e^{i\theta} = \cos \theta + i \sin \theta, \quad \text{for some } \theta \in \mathbb{R}.$$

Here θ is the *argument* of z (written $\text{Arg}(z)$), and it is the angle subtended by the ray from $(0, 0)$ to (a, b) (when $z = a + ib$, $a, b \in \mathbb{R}$), and the *real axis*.

S^1 is a metric space if the distance between two points $z, w \in S^1$ is defined to be the shortest distance between the two points, going around the circle. We define a map $f : S^1 \rightarrow S^1$ by $f(z) = z^2$. This map is called the *angle doubling map* because of the effect it has on $\theta = \text{Arg}(z)$: $f(e^{i\theta}) = e^{2i\theta}$. We see that the angle θ is doubled. It is clear that there are a lot of similarities between the doubling map and the angle doubling map, and we shall examine this in the next section when we study the notion of conjugacy for dynamical systems. First we show that the periodic points of f are dense in S^1 .

Consider the periodic points of $f(z) = z^2$: Solving $z^2 = z$ gives $z = 1$ (we can disregard $z = 0$), $f^2(z) = z$ gives $z^4 = z$ or $z^3 = 1$ and continuing in this way we see that the periodic points are certain n th roots of unity.

Proposition 5.1.3 *The periodic points of the angle doubling map $f : S^1 \rightarrow S^1$ are dense in S^1 .*

Proof. If $f^n(z) = z$ for some $n \in \mathbb{Z}^+$, then $z^{2^n} = z$ or $z^{2^n-1} = 1$. Write $z = e^{i\theta}$, then we want to find the $(2^n - 1)$ th roots of unity. This gives:

$$e^{(2^n-1)i\theta} = e^{2k\pi i}, \quad \text{for some } k \in \mathbb{Z}^+,$$

giving the $2^n - 1$ distinct roots: $z_k = e^{2k\pi i/(2^n-1)}$, $k = 0, 1, 2, \dots, 2^n - 2$, showing that

$$\text{Per}_n(f) = \{e^{2k\pi i/(2^n-1)} : 0 \leq k < 2^n - 1\}, \quad n \in \mathbb{Z}^+.$$

These points are equally spaced around the circle, a distance $2\pi/(2^n - 1)$ apart, which can be made arbitrarily small by taking n large enough. It follows that the periodic points are dense in S^1 . $\square$

5.2 Transitivity

Sometimes, given $f : X \rightarrow X$ (X a metric space), when we iterate $x_0 \in X$, the orbit $O(x_0) = \{x_0, f(x_0), \dots\}$, spreads itself evenly over X , so that $O(x_0)$ is a dense set in X . This leads to:

Definition 5.2.1 $f : X \rightarrow X$ is said to be (*topologically*) *transitive* if there exists $x_0 \in X$ such that $O(x_0)$ is a dense subset of X . A *transitive point* for f is a point x_0 which has a dense orbit under f . If f is transitive, then there is a dense set of transitive points, since each point in $O(x_0)$ will be a transitive point.

Example 5.2.2 The doubling map $D : [0, 1] \rightarrow [0, 1]$ is transitive: To show this we explicitly construct a point $x_0 \in [0, 1]$ which has a dense orbit under D . x_0 is defined using its binary expansion in the following way: first write down all possible “1-blocks”, i.e., 0 followed by 1. Then write down all possible “2-blocks”, i.e., 00, 01, 10, 11, then all possible “3-blocks”, i.e., 000, 001, 010, 011, 100, 101, 110, 111, and then continue in this way with all possible “4-blocks” etc. (we could write them down in the order in which they appear as in the binary expansion of the integers). This gives:

$$x_0 = \cdot 01\ 0001\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ 0000\ 0001\ \dots,$$

a point of $[0, 1]$.

To show that $O(x_0)$ is dense in $[0, 1]$, let $y \in [0, 1]$ with binary expansion

$$y = y_1 y_2 y_3 \dots = \sum_{i=1}^{\infty} \frac{y_i}{2^i}, \quad y_i = 0 \text{ or } 1,$$

and let $\delta > 0$.

Choose N so large that $\frac{1}{2^N} < \delta$. All possible finite strings of 0’s and 1’s appear in the binary expansion of x_0 , so the string $y_1 y_2 y_3 \dots y_N$ must also appear in the binary expansion of x_0 .

It follows that for some $r \in \mathbb{Z}^+$ we have

$$D^r(x_0) = \cdot y_1 y_2 y_3 \dots y_N b_{N+1} b_{N+2} \dots, \quad \text{for some } b_{N+1}, b_{N+2}, \dots,$$

so that

$$\begin{aligned} |D^r(x_0) - y| &= | \cdot y_1 y_2 \dots y_N b_{N+1} b_{N+2} \dots - \cdot y_1 y_2 \dots y_N y_{N+1} y_{N+2} \dots | \\ &\leq \sum_{i=N+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^N} < \delta. \end{aligned}$$

This shows that any point $y \in [0, 1]$ is arbitrarily close to the orbit of x_0 under D , so this orbit is dense in $[0, 1]$ $\square$

Remark 5.2.3 It is quite easy to see that for a transitive map $f : X \rightarrow X$ on a metric space, given any non-empty open sets U and V in X there exists $m \in \mathbb{Z}^+$ such that

$$U \cap f^m(V) \neq \emptyset.$$

Less easy to see is the converse of this statement, which holds for complete separable metric spaces (X is *separable* if it has a countable dense subset - see Chapter 7 for the notion of completeness). This is the *Birkhoff Transitivity Theorem*:

Theorem 5.2.4 *A continuous map f of a complete separable metric space X is transitive if and only if for every pair U and V of non-empty open subsets of X there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$.*

Using this result it is easy to show directly that the angle doubling map is transitive, however, the proof is beyond the scope of this text.

5.3 The Definition of Chaos

Devaney was the first to define the notion of chaos, saying that a function is chaotic if it has a dense set of periodic points, it is transitive and also has what is called *sensitive dependence on initial conditions* (known as the “butterfly effect” in the popular literature). Subsequently it was shown that the first two requirements imply the third, so we define chaos as follows:

Definition 5.3.1 Let $f : X \rightarrow X$ be a map of the metric space X , then f is said to be *chaotic* if:

- (i) The set of periodic points of f is dense in X .
- (ii) f is transitive.

Examples 5.3.2 1. Homeomorphisms or diffeomorphisms of an interval $I \subseteq \mathbb{R}$ cannot be chaotic as they are never transitive. The same type of considerations apply to functions such as $\sin x$, $\cos x$, $\arctan x$ and the logistic map L_μ for $0 < \mu < 3$.

2. We have shown that the doubling map $D : [0, 1] \rightarrow [0, 1]$ has a dense set of periodic points and is transitive, so it is a chaotic map.

3. The tent map $T : [0, 1] \rightarrow [0, 1]$ is chaotic. Recall that if $x \in [0, 1]$ has a binary expansion $x = \cdot a_1 a_2 a_3 \dots$, then $Tx = \begin{cases} \cdot a_2 a_3 a_4 \dots & \text{if } a_1 = 0 \\ \cdot a'_2 a'_3 a'_4 \dots & \text{if } a_1 = 1, \end{cases}$ where $a'_i = 1$ if $a_i = 0$ and $a'_i = 0$ if $a_i = 1$. More generally we can see using an induction argument that ([61]):

$$T^n x = \begin{cases} \cdot a_{n+1} a_{n+2} a_{n+3} \dots & \text{if } a_n = 0 \\ \cdot a'_{n+1} a'_{n+2} a'_{n+3} \dots & \text{if } a_n = 1. \end{cases}$$

We can use this to write down the periodic points of T . For example, the fixed points are $x = 0$ and $x = \cdot 1010\dots = 2/3$, and the period 2-points are

$$x_1 = \cdot 01100110\dots = 2/5 \quad \text{and} \quad x_2 = \cdot 11001100\dots = 4/5.$$

The period 3-points are

$$\cdot 010\,010\,010\,010\dots = 2/7, \quad \cdot 100\,100\,100\,100\dots = 4/7, \quad \cdot 110\,110\,110\,110\dots = 6/7$$

and

$$\cdot 001\,110\,001\,110\dots = 2/9, \quad \cdot 011\,100\,011\,100\dots = 4/9, \quad \cdot 111\,000\,111\,000\dots = 8/9.$$

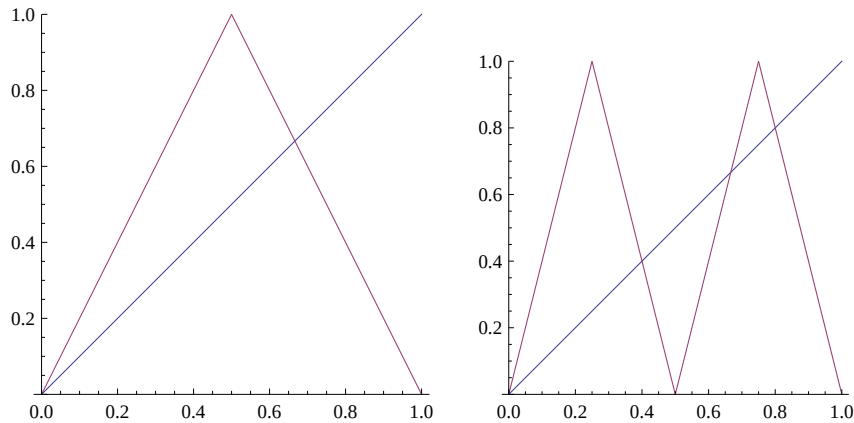
Notice that points of the form $x = k/2^n \in (0, 1)$, $k \in \mathbb{Z}^+$, are almost fixed since they have a binary expansion of the form

$$x = \cdot a_1 a_2 a_3 \dots, a_n 00 \dots,$$

where $a_n = 0$ if k is even and $a_n = 1$ if k is odd. It follows that $T^n x = 0$ if k is even and $T^n x = 1$ if k is odd. In particular

$$T^n \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right] = [0, 1].$$

The Intermediate Value Theorem now implies that there is a fixed point of T^n in the interval $[\frac{k-1}{2^n}, \frac{k}{2^n}]$. Since such intervals can be made arbitrarily small and cover all of $[0, 1]$, the set of periodic points must be dense in $[0, 1]$.



The fixed and period 2-points of the tent map.

We use these ideas to show that the periodic points of T are numbers in $[0, 1]$ of the form $x = r/s$ where r is an even integer and s is an odd integer.

Proof. If $x \in (0, 1)$ is a periodic point for T , then $T^n x = x$ for some $n \in \mathbb{Z}^+$. There are two cases to consider: Suppose that x has a binary expansion

$$x = \cdot a_1 a_2 a_3 \dots a_n a_{n+1} \dots, \quad \text{where } a_n = 0,$$

then

$$T^n x = \cdot a_{n+1} a_{n+2} \dots a_{2n} \dots,$$

so we must have $a_1 = a_{n+1}, a_2 = a_{n+2}, \dots, a_n = a_{2n} = 0$, and

$$x = \cdot a_1 a_2 \dots a_n a_1 a_2 \dots a_n a_1 \dots, \quad \text{where } a_n = 0,$$

Rewriting this gives

$$\begin{aligned} x &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_{n-1}}{2^{n-1}} \right) + \frac{1}{2^n} \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_{n-1}}{2^{n-1}} \right) + \dots \\ &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_{n-1}}{2^{n-1}} \right) \left(\frac{1}{1 - 1/2^n} \right) \\ &= \frac{a_1 2^{n-1} + a_2 2^{n-2} + \dots + a_{n-1} 2}{2^n - 1} = \frac{r}{s}, \end{aligned}$$

where r is even and s is odd.

In the second case $x = \cdot a_1 a_2 a_3 \dots, a_n a_{n+1} \dots$, where $a_n = 1$, so that

$$T^n x = \cdot a'_{n+1} a'_{n+2} \dots a'_{2n} \dots,$$

and this gives $a_1 = a'_{n+1}, a_2 = a'_{n+2}, \dots, a_n = a'_{2n}, \dots$, and

$$x = \cdot a_1 a_2 \dots a_{n-1} 1 a'_1 a'_2 \dots a'_{n-1} 0 a_1 \dots$$

We can now argue as before, but using the first $2n$ terms of x .

Conversely, suppose that $x = r/s \in (0, 1)$ where r is an even integer and s is an odd integer. Since s and 2 are coprime, we can apply Euler's generalization of Fermat's Theorem to give

$$2^{\phi(s)} \equiv 1 \pmod{s}, \quad \text{or} \quad 2^p - 1 = ks \quad \text{for some } p, k \in \mathbb{Z}^+,$$

($1 < kr < 2^p - 1$), where ϕ is Euler's function. Write the binary expansion of kr as

$$kr = a_1 2^{p-1} + a_2 2^{p-2} + \dots + a_{p-2} 2^2 + a_{p-1} 2, \quad a_i \in \{0, 1\},$$

which is even, so

$$\begin{aligned} \frac{kr}{2^p - 1} &= (a_1 2^{p-1} + a_2 2^{p-2} + \dots + a_{p-2} 2^2 + a_{p-1} 2) \left(\frac{1/2^p}{1 - 1/2^p} \right) \\ &= \left(\frac{a_1}{2} + \frac{a_2}{2^2} + \dots + \frac{a_{p-1}}{2^{p-1}} \right) \left(\frac{1}{1 - 1/2^p} \right) \\ &= \cdot a_1 a_2 \dots a_{p-1} 0 a_1 a_2 \dots a_{p-1} 0 \dots, \end{aligned}$$

which is a point of period p (or less). □

It is now clear that T is transitive since if we define $x_0 \in (0, 1)$ having a binary expansion consisting of all 1-blocks, all 2-blocks, all 3-blocks etc., as before, except that we insert a single zero between every block then it follows that if $x = \cdot x_1 x_2 \dots x_n x_{n+1} \dots$, then $T^p x_0 = \cdot x_1 x_2 \dots x_n b_{n+1} \dots$ for some $p > 0$, i.e., every block will appear in the iterates of x_0 . As before we see that T is transitive, so is chaotic.

4. It is possible for a map to be transitive without being chaotic (although for continuous functions f on intervals in $\mathbb{R}$, this is not possible: see [64]). For example, consider the *irrational rotation* $R_a : S^1 \rightarrow S^1$ defined by $R_a(z) = a \cdot z$ for some (fixed) $a \in S^1$. To say that R_a is an irrational rotation means that $a^n \neq 1$ for any $n \in \mathbb{Z}^+$, i.e., a is not an n th root of unity for any $n \in \mathbb{Z}^+$. It can be shown that *every* $z_0 \in S^1$ has a dense orbit (a transformation with this property is said to be *minimal*). However, suppose that $R_a^n(z) = z$, then $a^n z = z$ or $a^n = 1$, a contradiction, so that R_a has no periodic points. R_a is an example of an *isometry*: points always stay the same distance apart:

$$|R_a(z) - R_a(w)| = |az - aw| = |a||z - w| = |z - w|.$$

Note that if instead we have $a^n = 1$ for some $n \in \mathbb{Z}^+$, then $R_a^n(z) = a^n z = z$ for all $z \in S^1$, so that R_a^n is just the *identity map* (every point of S^1 is of period n).

5.4 Some Symbolic Dynamics and the shift map

Recall that the set of all infinite sequences of 0's and 1's:

$$\Sigma = \{\omega = (s_1, s_2, s_3, \dots) : s_i = 0 \text{ or } 1\},$$

is a metric space with metric defined by

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k}, \text{ where } \omega_1 = (s_1, s_2, \dots), \omega_2 = (t_1, t_2, \dots) \in \Sigma.$$

This metric has the following properties:

Property 5.4.1 If $\omega_1 = (s_1, s_2, \dots)$, $\omega_2 = (t_1, t_2, \dots) \in \Sigma$, with $s_i = t_i$, $i = 1, 2, \dots, n$, then $d(\omega_1, \omega_2) \leq 1/2^n$.

Proof.

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k} = \sum_{k=n+1}^{\infty} \frac{|s_k - t_k|}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

□

Property 5.4.2 If $d(\omega_1, \omega_2) < 1/2^n$, then $s_i = t_i$ for $i = 1, 2, \dots, n$.

Proof. We give a proof by contradiction: Suppose that $s_j = t_j$ for some $1 \leq j \leq n$, then

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{|s_k - t_k|}{2^k} \geq \frac{1}{2^j} \geq \frac{1}{2^n},$$

a contradiction. □

The shift map σ (sometimes called the *Bernoulli shift*) is an important function defined on Σ .

Definition 5.4.3 The shift map $\sigma : \Sigma \rightarrow \Sigma$ is defined by

$$\sigma(s_1, s_2, s_3, \dots) = (s_2, s_3, \dots),$$

so for example $\sigma(1, 0, 1, 0, \dots) = (0, 1, 0, 1, \dots)$ and $\sigma^2(1, 0, 1, 0, \dots) = (1, 0, 1, 0, \dots)$, so that if $\omega_1 = (1, 0, 1, 0, \dots)$ and $\omega_2 = (0, 1, 0, 1, \dots)$, then $\{\omega_1, \omega_2\}$ is a 2-cycle for σ . In this way, it is easy to write down all of the points of period n . Any sequence which is eventually constant is clearly an eventually fixed point of Σ , and any sequence which is eventually periodic (such as $(1, 1, 1, 0, 1, 0, 1, 0, 1, \dots)$), is an eventually periodic point.

Proposition 5.4.4 The shift map $\sigma : \Sigma \rightarrow \Sigma$ is continuous, onto, but not one-to-one.

Proof. Clearly σ is onto but not one-to-one. To show that σ is continuous, let $\epsilon > 0$, then we want to find $\delta > 0$ such that if

$$d(\omega_1, \omega_2) < \delta, \quad \text{then} \quad d(\sigma(\omega_1), \sigma(\omega_2)) < \epsilon.$$

We shall see that it suffices to take $\delta = 1/2^{n+1}$ if n is chosen so large that $1/2^n < \epsilon$.

In this case, if $d(\omega_1, \omega_2) < \delta = 1/2^{n+1}$, then from Property 5.4.2, $s_i = t_i$ for $i = 1, 2, \dots, n+1$. Clearly the first n terms of the sequences $\sigma(\omega_1)$ and $\sigma(\omega_2)$ are equal, so by Property 5.4.1, $d(\sigma(\omega_1), \sigma(\omega_2)) \leq 1/2^n < \epsilon$, so that σ is continuous. □

We can now prove:

Theorem 5.4.5 The shift map $\sigma : \Sigma \rightarrow \Sigma$ is chaotic.

Proof. We first show that the periodic points are dense in Σ . Let $\omega = (s_1, s_2, \dots) \in \Sigma$, then it suffices to show that there is a sequence of periodic points $\omega_n \in \Sigma$ with $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$. Set

$$\omega_1 = (s_1, s_1, s_1, s_1, \dots), \text{ a period 1-point for } \sigma,$$

$\omega_2 = (s_1, s_2, s_1, s_2, \dots)$, a period 2-point for σ ,

$\omega_3 = (s_1, s_2, s_3, s_1, s_2, s_3, \dots)$, a period 3-point for σ ,

and continuing in this way so that

$\omega_n = (s_1, s_2, \dots, s_n, s_1, \dots)$, a period n -point for σ .

Since ω and ω_n agree in the first n coordinates, $d(\omega, \omega_n) \leq 1/2^n$, so $d(\omega, \omega_n) \rightarrow 0$ as $n \rightarrow \infty$, or $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$.

To show that σ is transitive, we explicitly construct a point $\omega_0 \in \Sigma$ having a dense orbit under σ . Set

$$\omega_0 = (\underbrace{01}_{1\text{-blocks}} \underbrace{00\ 01\ 10\ 11}_{\substack{\text{all possible} \\ 2\text{-blocks}}} \underbrace{000\ 001\ 010\ 101\ 011\ \dots}_{\text{all possible 3 blocks}}),$$

and continuing in this way so that all possible n -blocks appear in ω_0 . To see that $\overline{O(\omega_0)} = \Sigma$, let $\omega = (s_1, s_2, s_3, \dots) \in \Sigma$ be arbitrary. Let $\epsilon > 0$ and choose n so large that $1/2^n < \epsilon$, then since ω_0 consists of all possible n -blocks, the sequence $(s_1, s_2, \dots, s_n)$ must appear somewhere in ω_0 , i.e., there exists $k > 0$ with

$$\sigma^k(\omega_0) = (s_1, s_2, \dots, s_n, \dots),$$

so that ω and $\sigma^k(\omega_0)$ agree on the first n coordinates. It follows that

$$d(\omega, \sigma^k(\omega_0)) \leq \frac{1}{2^n} < \epsilon.$$

This shows that the orbit of ω_0 comes arbitrarily close to any member of Σ , so it is dense in Σ . It follows that σ is transitive and so it is chaotic. $\square$

Remark 5.4.6 It can be shown that (topological) properties such as being totally disconnected, perfect etc. are preserved by homeomorphisms.

The shift space Σ and the Cantor set C are homeomorphic metric spaces. In addition, the half open interval $[0, 1)$ and the unit circle S^1 in the complex plane are homeomorphic. In particular, Σ and C will have identical topological properties, and so will S^1 and $[0, 1)$. It follows that Σ and $[0, 1]$ cannot be homeomorphic as C and $[0, 1]$ are not homeomorphic (C is totally disconnected, but $[0, 1]$ is not).

Proof. Σ is given its usual metric, and C has the metric induced from being a subset of $\mathbb{R}$, so that $d(x, y) = |x - y|$ for $x, y \in C$.

We define a map $h : C \rightarrow \Sigma$ by $h(a_1 a_2 a_3 \dots) = (s_1, s_2, s_3, \dots)$, where $a_i = 0$ or 2 and $s_i = a_i/2$. Clearly h is both one-to-one and onto. We show that it is continuous at each $x_0 \in C$.

Let $\epsilon > 0$ and choose n so large that $1/2^n < \epsilon$. Set $\delta = 1/3^n$, then if $|x_0 - x| < \delta$, both x_0 and x must lie in the same component (sub-interval) of S_n of length $1/3^n$. It follows that x_0 and x must have an identical ternary expansions in the first n places. Correspondingly, $h(x_0)$ and $h(x)$ must have the same first n coordinates. It follows that $d(h(x_0), h(x)) \leq 1/2^n < \epsilon$, so h is continuous at x_0 . In a similar way we see that h^{-1} is continuous.

To see that $[0, 1)$ and S^1 are homeomorphic, define $h : [0, 1) \rightarrow S^1$ by $h(x) = e^{2\pi i x}$, then h is clearly one-to-one and onto. The map h wraps the interval $[0, 1)$ around the circle, so that strictly speaking we are looking at $[0, 1]$ with the end points identified, so in this way h becomes continuous. $\square$

Exercises 5.4

1. Let $f : X \rightarrow X$ be a transitive map of the metric space (X, d) . Show that if U and V are non-empty open sets in X , there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$.
2. Let $F : [0, 1) \rightarrow [0, 1)$ be the tripling map $F(x) = 3x \bmod 1$. Follow the proof for the doubling map (but use ternary expansions) to show that F is transitive and the period points are dense (find the periodic points), and hence show that F is chaotic.
3. Let $D : [0, 1) \rightarrow [0, 1)$ be the doubling map. Show that $|\text{Per}_n(D)| = 2^n$.
4. Use Proposition 4.4.7 to show that for a continuous increasing function $f : [a, b] \rightarrow [a, b]$, the periodic points cannot be dense in $[a, b]$ (so a homeomorphism of $[a, b]$ cannot be chaotic).

5.5 Sensitive Dependence on Initial Conditions

We now show that the original definition of chaos due to Devaney [20] follows from the definition we have given. This result is due to Banks et al. [6].

Definition 5.5.1 Let $f : X \rightarrow X$ be defined on a metric space (X, d) . Then f has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any $x \in X$ and any open interval U containing x and points other than x , there is a point $y \in U$ and $n \in \mathbb{Z}^+$ with

$$d(f^n(x), f^n(y)) > \delta.$$

This is the precise definition of the idea that iterates of points close to each other may eventually be widely apart, so that a map has sensitive dependence on initial conditions if there exist points arbitrarily close to x which are eventually at least distance δ away from x . It is important to know whether we have sensitive dependence when doing computations as round-off errors may be magnified after numerous iterations. For example, suppose we iterate the doubling map, starting with $x_0 = 1/3$ and $x_1 = .333$. After 10 iterations we have $D^{10}(x_0) = 1/3$ and $D^{10}(x_1) = .92$, more than distance $1/2$ apart.

Examples 5.5.2 1. The linear map $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = ax$, $|a| > 1$, has sensitive dependence since if $x \neq y$,

$$|f^n(x) - f^n(y)| = |a^n x - a^n y| = a^n |x - y| \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

However, clearly the dynamics of f is not complicated (f is not chaotic).

2. The shift map $\sigma : \Sigma \rightarrow \Sigma$ has sensitive dependence on initial conditions since if $\omega_1, \omega_2 \in \Sigma$ with $\omega_1 \neq \omega_2$, then they must differ at some coordinates, say $s_i \neq t_i$. Then

$$\sigma^{i-1}(\omega_1) = (s_i, s_{i+1}, \dots), \quad \text{and} \quad \sigma^{i-1}(\omega_2) = (t_i, t_{i+1}, \dots),$$

so that

$$d(\sigma^{i-1}(\omega_1), \sigma^{i-1}(\omega_2)) = \sum_{k=1}^{\infty} \frac{|s_{i+k-1} - t_{i+k-1}|}{2^k} = \frac{1}{2} + \text{other terms} \geq \frac{1}{2}.$$

3. The angle doubling map $f : S^1 \rightarrow S^1$, $f(z) = z^2$ has sensitive dependence since if we iterate $z = e^{i\theta}, w = e^{i\phi} \in S^1$, their distance apart doubles after each iteration.

4. The doubling map $D : [0, 1] \rightarrow [0, 1]$ can be seen directly to have sensitive dependence on initial conditions. This also follows from the following theorem:

Theorem 5.5.3 (Banks et al. [6]) *Let $f : X \rightarrow X$ be a chaotic transformation, then f has sensitive dependence on initial conditions.*

We first prove a preliminary result:

Lemma 5.5.4 *Let $f : X \rightarrow X$ be a transformation which has at least two different periodic orbits. Then there exists $\epsilon > 0$ such that for any $x \in X$ there is a periodic point p satisfying*

$$d(x, f^k(p)) > \epsilon, \quad \text{for all } k \in \mathbb{Z}^+.$$

Proof. Let a and b be two periodic points with different orbits. Then $d(f^k(a), f^l(b)) > 0$ for all k and l (since we are dealing with finite sets).

Choose $\epsilon > 0$ small enough that $d(f^k(a), f^l(b)) > 2\epsilon$ for all k and l . Then

$$d(f^k(a), x) + d(x, f^l(b)) \geq d(f^k(a), f^l(b)) > 2\epsilon \quad \forall k, l \in \mathbb{Z}^+,$$

by the triangle inequality.

If x is within ϵ of any of the points $f^l(b)$, then it must be at a greater distance than ϵ from all of the points $f^k(a)$ and the result follows. $\square$

Proof of Theorem 5.5.3 Let $x \in X$ and U be an open set in X containing x .

Let p be a periodic point of period r for f , whose orbit is a distance greater than 4δ from x .

The periodic points of f are dense in X , so there is a periodic point q of period n say, with

$$q \in V = U \cap B_\delta(x).$$

Write

$$W_i = B_\delta(f^i(p)),$$

then

$$f^i(p) \in W_i, \forall i \Rightarrow p \in f^{-i}(W_i), \forall i,$$

so the open set

$$W = f^{-1}(W_1) \cap f^{-2}(W_2) \cap \cdots \cap f^{-n}(W_n) \neq \emptyset.$$

Since f is transitive, there is a point $z \in V$ with $f^k(z) \in W$ for some $k \in \mathbb{Z}^+$.

Let j be the smallest integer with $k < nj$, or

$$1 \leq nj - k \leq n.$$

Then

$$f^{nj}(z) = f^{nj-k}(f^k(z)) \in f^{nj-k}(W).$$

But

$$f^{nj-k}(W) = f^{nj-k}(f^{-1}(W_1) \cap f^{-2}(W_2) \cap \cdots \cap f^{-n}(W_n)) \subset f^{nj-k}(f^{-(nj-k)}W_{nj-k}) = W_{nj-k},$$

so that $d(f^{nj}(z), f^{nj-k}(p)) < \delta$. Now $f^{nj}(q) = q$, and by the triangle inequality

$$d(f^{nj-k}(p), x) \leq d(f^{nj-k}(p), f^{nj}(z)) + d(f^{nj}(z), f^{nj}(q)) + d(f^{nj}(q), x)$$

so that

$$\begin{aligned} 4\delta < d(f^{nj-k}(p), x) &\leq d(f^{nj-k}(p), f^{nj}(z)) + d(f^{nj}(z), f^{nj}(q)) + d(q, x) \\ &< \delta + d(f^{nj}(z), f^{nj}(q)) + \delta, \end{aligned}$$

so

$$d(f^{nj}(z), f^{nj}(q)) > 2\delta.$$

This inequality implies that either

$$d(f^{nj}(x), f^{nj}(z)) \geq \delta,$$

or

$$d(f^{nj}(x), f^{nj}(q)) \geq \delta,$$

for if $f^{nj}(x)$ were within distance $< \delta$ from both of these points, they would have to be within $< 2\delta$ from each other, contradicting the previous inequality above. So one of the two, z or q will serve as the y in the theorem with $m = nj$. $\square$

Chapter 6. Conjugacy of Dynamical Systems

Two metric spaces X and Y are the “same” (homeomorphic) if there is a homeomorphism from one space to the other. In this chapter we study the question of when two *dynamical systems* are the same. Given maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$, we require them to have the same type of dynamical behavior, e.g., if one is chaotic, then so is the other, there is a one-to-one correspondence between their periodic points etc. One obvious requirement is that the underlying metric spaces should be homeomorphic. We have seen a lot of similarities between the logistic map $L_4(x) = 4x(1 - x)$ and the tent map $T(x)$ and this will be examined in this chapter together with other examples such as the shift map and circle maps.

6.1 Conjugate Maps

This “sameness” is given by the idea of conjugacy, a notion borrowed from group theory, where two members a and b of a group G are conjugate if there exists $g \in G$ with $ag = gb$. One of the central problems of one-dimensional dynamics and dynamical systems in general is to be able to tell whether or not two dynamical systems are conjugate. We will see that if one map has a 3-cycle and another map has no 3-cycle (for example), then the maps cannot be conjugate, or if one map has 2 fixed points and the other has 3 fixed points, then they are not conjugate. These are examples of *conjugacy invariants*, which give criteria for maps to be non-conjugate. A generally harder problem is establishing a conjugacy between maps.

Definition 6.1.1 1. Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be maps of metric spaces. Then f and g are said to be *conjugate* if there is a homeomorphism $h : X \rightarrow Y$ such that

$$h \circ f = g \circ h.$$

The map h is a *conjugacy* between f and g . Obviously conjugacy is an equivalence relation.

2. If in the above definition we only require the map $h : X \rightarrow Y$ to be continuous, then we say that g is a *factor* of f . If in addition, h is an onto map, then we say that g is a *quasi-factor* of f .

Exercises 6.1

1. Prove that if $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are conjugate maps of metric spaces, then f is one-to-one if and only if g is one-to-one, and f is onto if and only if g is onto.

2. Prove that if f and g are conjugate via h and f has a local maximum at x_0 , then g has a local maximum or minimum at $h(x_0)$.
3. Suppose that $h : [0, 1] \rightarrow [0, 1]$ is a conjugacy between $f, g : [0, 1] \rightarrow [0, 1]$ where $f(0) = f(1) = 0$ and $g(0) = g(1) = 0$. Show that h is increasing on $[0, 1]$. Deduce that h maps the zero's of f to the zero's of g .
4. ([11]) The function $T_n(x) = \cos(n \arccos(x))$ is the n th Chebyshev polynomial. Show that T_n is conjugate to the map $\Lambda_n : [0, 1] \rightarrow [0, 1]$, the piecewise linear continuous map defined by joining the points $(0, 0), (1/n, 1), (2/n, 0), (3/n, 1), \dots$, ending with $(1, 1)$ if n is odd, or $(1, 0)$ if n is even. Use the conjugacy map $h : [0, 1] \rightarrow [0, 1]$, $h(x) = \cos(\pi x)$. (In [11], there is a generalization of this to maps T_λ , where $\lambda > 1$ is a real number).

6.2 Properties of Conjugate Maps

It is often easier to show that certain maps are chaotic indirectly by showing that they are conjugate to chaotic maps and using the following result:

Proposition 6.2.1 *If $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are maps conjugate via a conjugacy $h : X \rightarrow Y$: $h \circ f = g \circ h$, then*

1. $h \circ f^n = g^n \circ h$ for all $n \in \mathbb{Z}^+$, (so f^n and g^n are also conjugate).
2. If c is a point of period m for f , then $h(c)$ is a point of period m for g . c is attracting if and only if $h(c)$ is attracting.
3. f is transitive if and only if g is transitive.
4. f has a dense set of periodic points if and only if g has a dense set of periodic points.
5. f is chaotic if and only if g is chaotic.

Proof. 1. $h \circ f^2 = h \circ f \circ f = g \circ h \circ f = g \circ g \circ h = g^2 \circ h$, and in the same way $h \circ f^3 = g^3 \circ h$, and continuing inductively the result follows.

2. Suppose that $f^i(c) \neq c$ for $0 < i < m$ and $f^m(c) = c$, then $h \circ f^i(c) \neq h(c)$ for $0 < i < m$ since h is one-to-one, and so $g^i \circ h(c) \neq h(c)$ for $0 < i < m$. In addition, $h \circ f^m(c) = g^m \circ h(c)$, or $h(c) = g^m(h(c))$, so $h(c)$ is a period m point for g .

We shall only show that if p is an attracting fixed point of f (so that there is an open ball $B_\epsilon(p)$ such that if $x \in B_\epsilon(p)$ then $f^n(x) \rightarrow p$ as $n \rightarrow \infty$), then $h(p)$ is an attracting fixed point of g .

Let $V = h(B_\epsilon(p))$, then since h is a homeomorphism, V is open in Y and contains $h(p)$. Let $y \in V$, then $h^{-1}(y) \in B_\epsilon(p)$, so that $f^n(h^{-1}(y)) \rightarrow p$ as $n \rightarrow \infty$.

Since h is continuous, $h(f^n(h^{-1}(y))) \rightarrow h(p)$ as $n \rightarrow \infty$, i.e.,

$$g^n(y) = h \circ f^n \circ h^{-1}(y) \rightarrow h(p), \quad \text{as } n \rightarrow \infty,$$

so $h(p)$ is attracting.

3. Suppose that $O(z) = \{z, f(z), f^2(z), \dots\}$ is dense in X and let $V \subset Y$ be a non-empty open set. Then since h is a homeomorphism, $h^{-1}(V)$ is open in X , so there exists $k \in \mathbb{Z}^+$ with $f^k(z) \in h^{-1}(V)$.

It follows that $h(f^k(z)) = g^k(h(z)) \in V$, so that

$$O(h(z)) = \{h(z), g(h(z)), g^2(h(z)), \dots\}$$

is dense in Y , i.e., g is transitive. Similarly, if g is transitive, then f is transitive.

4. Suppose that f has a dense set of periodic points and let $V \subset Y$ be non-empty and open. Then $h^{-1}(V)$ is open in X , so contains periodic points of f . As in (3), we see that V contains periodic points of g . Similarly if g has a dense set of periodic points, so does f .

5. This now follows from (3) and (4). □

Example 6.2.2 We remark that sensitive dependence on initial conditions is not a conjugacy invariant. It is possible for two maps on metric spaces to be conjugate, one to have sensitive dependence, but the other not: Consider $T : (0, \infty) \rightarrow (0, \infty)$, $T(x) = 2x$ and $S : \mathbb{R} \rightarrow \mathbb{R}$ defined by $S(x) = x + \ln 2$. If $H : (0, \infty) \rightarrow \mathbb{R}$ is defined by $H(x) = \ln x$, then H is a homeomorphism and we can check that $H \circ T = S \circ H$ so T and S are conjugate, T has sensitive dependence, but S does not.

It can be shown however, that if $T : X \rightarrow X$ is a map on a compact metric space X (for example $X = [0, 1]$) having sensitive dependence, then any map conjugate to T also has sensitive dependence.

It can also be shown that the property of having negative Schwarzian derivative is not a conjugacy invariant.

Example 6.2.3 1. The logistic map $L_4 : [0, 1] \rightarrow [0, 1]$, $L_4(x) = 4x(1-x)$ is conjugate to the tent map $T : [0, 1] \rightarrow [0, 1]$, $T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{if } 1/2 < x \leq 1 \end{cases}$.

Proof. Define $h : [0, 1] \rightarrow [0, 1]$ by $h(x) = \sin^2(\pi x/2)$. We can see that h is one-to-one, onto and both h and h^{-1} are continuous, so it is a homeomorphism (it is not a diffeomorphism as $h'(1) = 0$). Also

$$L_4 \circ h(x) = L_4\left(\sin^2\left(\frac{\pi x}{2}\right)\right) = 4\sin^2\left(\frac{\pi x}{2}\right)\left(1 - \sin^2\left(\frac{\pi x}{2}\right)\right) = \sin^2(\pi x),$$

and

$$h \circ T(x) = h(Tx) = \begin{cases} h(2x) & \text{if } 0 \leq x \leq 1/2 \\ h(2-2x) & \text{if } 1/2 < x \leq 1 \end{cases} = \sin^2(\pi x),$$

so $L_4 \circ h = h \circ T$ and L_4 and T are conjugate. $\square$

2. The doubling map $D : [0, 1] \rightarrow [0, 1]$ is a quasi-factor of the shift map $\sigma : \Sigma \rightarrow \Sigma$.

Proof. D is a factor of σ since

$$h \circ \sigma = D \circ h,$$

where $h : \Sigma \rightarrow [0, 1]$ is defined by

$$h(x_1, x_2, x_3, \dots) = .x_1 x_2 x_3 \dots = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$$

is a continuous function (note that h is not a homeomorphism since it is not one-to-one: for example $h(1, 0, 0, \dots) = 1/2 = h(0, 1, 1, 1, \dots)$, but $(1, 0, 0, \dots) \neq (0, 1, 1, 1, \dots)$ in Σ). In addition $h(\Sigma) = [0, 1]$, so that h is onto and D is a quasi-factor of σ $\square$

4. The logistic map L_4 is a quasi-factor of the angle doubling map $f : S^1 \rightarrow S^1$, $f(z) = z^2$.

Proof. Define $h : S^1 \rightarrow [0, 1]$ by $h(e^{ix}) = \sin^2 x$, then

$$L_4 \circ h(e^{ix}) = L_4(\sin^2 x) = 4\sin^2(1 - \sin^2 x) = \sin^2(2x),$$

and

$$h \circ f(e^{ix}) = h(e^{2ix}) = \sin^2(2x).$$

h is clearly onto and continuous, but it is not one-to-one: $h(e^{ix}) = h(e^{-ix})$, so L_4 is a quasi-factor of f , but h is not a conjugacy. $\square$

We can now show that many of the above maps are chaotic. In order to do this we need to weaken the conditions of Proposition 6.2.1. If we drop the requirement that h is a homeomorphism, but just require it to be continuous and onto, then we can show that if f is chaotic then so is g . In other words, if g is a quasi-factor of f where f is chaotic, then g is also chaotic. This result will be useful in showing that a number of well known examples are chaotic.

Proposition 6.2.4 *Let $h : X \rightarrow Y$ be surjective (onto) and continuous. If $f : X \rightarrow X$ and $g : Y \rightarrow Y$ satisfy $h \circ f = g \circ h$ and f is chaotic, then g is chaotic.*

Before proving this proposition we need a lemma concerning continuous functions on metric spaces:

Lemma 6.2.5 *Let $h : X \rightarrow Y$ be a continuous function of metric spaces and A a subset of X , then $h(\overline{A}) \subseteq \overline{h(A)}$.*

Proof. Let $y \in h(\overline{A})$, then there exists $x \in \overline{A}$ with $y = h(x)$. We can find a sequence $x_n \in A$ with $\lim_{n \rightarrow \infty} x_n = x$.

Then $h(x_n) \in h(A)$ and since h is continuous

$$\lim_{n \rightarrow \infty} h(x_n) = h(x) = y \quad \text{so that} \quad y \in \overline{h(A)}.$$

□

Proof of Proposition 6.2.4 Use $\text{Per}(f)$ and $\text{Per}(g)$ to denote the periodic points of f and g respectively, then we saw earlier that $h(\text{Per}(f)) \subseteq \text{Per}(g)$. Since f is chaotic, $\overline{\text{Per}(f)} = X$, and since h is onto, $h(X) = Y$. Then using the lemma we have

$$Y = h(X) = h(\overline{\text{Per}(f)}) \subseteq \overline{h(\text{Per}(f))} \subseteq \overline{\text{Per}(g)},$$

so that $\overline{\text{Per}(g)} = Y$. In other words, the periodic points of g are dense in Y .

f is transitive so there exists $x_0 \in X$ with $\overline{O_f(x_0)} = X$ (where we use the subscript to distinguish the orbits with respect to f and g). Now

$$\begin{aligned} h(O_f(x_0)) &= h\{f^n(x_0) : n \in \mathbb{Z}^+\} = \{h \circ f^n(x_0) : n \in \mathbb{Z}^+\} \\ &= \{g^n \circ h(x_0) : n \in \mathbb{Z}^+\} = O_g(h(x_0)), \end{aligned}$$

so that

$$Y = h(X) = h(\overline{O_f(x_0)}) \subseteq \overline{h(O_f(x_0))} = \overline{O_g(h(x_0))},$$

so that $h(x_0)$ is a transitive point for g .

□

It is easily seen that Proposition 6.3.4 remains true if we replace the requirement that h be onto by requiring that $h(X)$ be dense in Y .

Theorem 6.2.6 *The tent map $T : [0, 1] \rightarrow [0, 1]$, the logistic map $L_4(x) = 4x(1 - x)$, the angle doubling map $f : S^1 \rightarrow S^1$, $f(z) = z^2$, and the Doubling map $D : [0, 1] \rightarrow [0, 1]$ are all chaotic.*

Proof. Suppose that we can show that the angle-doubling map f is a quasi-factor of the doubling map D , then we have:

The tent map T is conjugate to the logistic map L_4 , which is a quasi-factor of $f(z)$, which is quasi-factor of D , which is a quasi-factor of σ , the shift map. It has been shown that the shift map is chaotic. The result now follows.

It therefore suffices to show that for $f : S^1 \rightarrow S^1$, $f(z) = z^2$ and $D : [0, 1] \rightarrow [0, 1]$, $D(x) = 2x \pmod{1}$, f is a quasi-factor of D . Define $h : [0, 1] \rightarrow S^1$ by $h(x) = e^{2\pi i x}$, then h is continuous, onto and one-to-one everywhere except that $h(0) = 1 = h(1)$. Now

$$f \circ h(x) = f(e^{2\pi i x}) = e^{4\pi i x},$$

and

$$h \circ D(x) = h(2x \pmod{1}) = e^{2\pi i (2x \pmod{1})} = e^{4\pi i x},$$

or $h \circ D = f \circ h$, and f is a quasi-factor of D . □

Exercises 6.2

1. If $g(z) = z^3$ on S^1 , show that g is the angle tripling map. Find the period points and show that they are dense in S^1 . Show that g is conjugate to the map $F : [0, 1) \rightarrow [0, 1)$, $F(x) = 3x \pmod{1}$ of Exercises 5.4.

1. If $T_3 : C \rightarrow C$ is the tent map with $\mu = 3$, but restricted to the Cantor set. Show that T_3 is conjugate to the shift map $\sigma : \Sigma \rightarrow \Sigma$.

3. Prove that the doubling map $D : [0, 1) \rightarrow [0, 1)$, $D(x) = 2x \pmod{1}$, and the angle doubling map $f : S^1 \rightarrow S^1$, $f(z) = z^2$, are conjugate.

4. Is the shift map $\sigma : \Sigma \rightarrow \Sigma$ conjugate to the doubling map D ? (It can be shown that the shift map σ is conjugate to T_3 restricted to the Cantor set).

5. Let $U : [-1, 1] \rightarrow [-1, 1]$ be defined by $U(x) = 1 - 2x^2$ and $T_2 : [0, 1] \rightarrow [0, 1]$ be the tent map

$$T_2(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2(1-x) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Prove that $H : [0, 1] \rightarrow [-1, 1]$, $H(x) = -\cos(\pi x)$ defines a conjugacy between these maps.

6. Prove that the map $F : [-1, 1] \rightarrow [-1, 1]$, $F(x) = 4x^3 - 3x$ is conjugate to $T : [0, 1] \rightarrow [0, 1]$, $T(x) = 3x \bmod(1)$, via the conjugacy $h : [0, 1] \rightarrow [-1, 1]$, $h(x) = \cos(\pi x)$. (Hint: $\cos(3x) = 4\cos^3(x) - 3\cos(x)$). Deduce that F is chaotic. This question is related to Exercise 6.1, # 4 concerning the Chebyshev polynomials.

6.3 Linear Conjugacy

It is sometimes the case that the conjugacy between two real (or complex) functions is given by a map with a straight line graph (an affine map). This is called a linear conjugacy, and is stronger than the usual notion of conjugacy.

Definition 6.3.1 For functions $f : I \rightarrow I$ and $g : J \rightarrow J$ defined on subintervals of $\mathbb{R}$, we say that f and g are *linearly conjugate* and that h is a *linear conjugacy* if h maps I onto J where $h(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$ and $h \circ f = g \circ h$.

The following example gives a criterion for two quadratic functions to be linearly conjugate:

Example 6.3.2 Let

$$F(x) = ax^2 + bx + c \quad \text{and} \quad G(x) = rx^2 + sx + t,$$

where $a \neq 0$ and $r \neq 0$. If

$$c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a},$$

then F and G are linearly conjugate via the linear conjugacy

$$h(x) = \frac{a}{r}x + \frac{b-s}{2r}.$$

Proof.

$$\begin{aligned} h \circ F(x) &= h(ax^2 + bx + c) = \frac{a(ax^2 + bx + c)}{r} + \frac{b-s}{2r} \\ &= \frac{a^2}{r}x^2 + \frac{ab}{r}x + \frac{2ac + b-s}{2r}, \end{aligned}$$

and

$$\begin{aligned} G \circ h(x) &= G\left(\frac{a}{r}x + \frac{b-s}{2r}\right) = r\left(\frac{a}{r}x + \frac{b-s}{2r}\right)^2 + s\left(\frac{a}{r}x + \frac{b-s}{2r}\right) + t \\ &= r\left(\frac{a^2}{r^2}x^2 + 2\frac{a(b-s)}{2r^2}x + \frac{(b-s)^2}{4r^2}\right) + \frac{sa}{r}x + \frac{bs-s^2}{2r} + t \\ &= \frac{a^2}{r}x^2 + \frac{ab}{r}x + \frac{(b-s)^2 + 2bs - 2s^2 + 4rt}{4r}, \end{aligned}$$

and we see that these are equal if

$$c = \frac{b^2 - s^2 + 2s - 2b + 4rt}{4a}.$$

□

For example, if F is defined on the interval $[0, 1]$, then

$$h(0) = \frac{b-s}{2r} \quad \text{and} \quad h(1) = \frac{2a+b-s}{2r},$$

so if $a/r > 0$, then F is conjugate to G on the interval $[\frac{b-s}{2r}, \frac{2a+b-s}{2r}]$.

Example 6.3.3 1. If $L_\mu(x) = \mu x(1-x)$, and $Q_c(x) = x^2 + c$ and $c = \frac{2\mu - \mu^2}{4}$, then L_μ on the interval $[0, 1]$ is linearly conjugate to Q_c on the interval $[-\mu/2, \mu/2]$. If $\mu = 4$, this shows that $L_4(x) = 4x(1-x)$ on $[0, 1]$ is conjugate to $Q_c(x) = x^2 + c$ on the interval $[-2, 2]$ when $c = -2$. In particular, Q_{-2} on $[-2, 2]$ is chaotic.

Proof. We apply Example 6.3.2 with

$$a = -\mu, \quad b = \mu, \quad c = 0, \quad r = 1, \quad s = 0, \quad t = c.$$

In this case $h(0) = \mu/2$ and $h(1) = -\mu/2$ and we can check that the conditions of the example hold when $c = \frac{2\mu - \mu^2}{4}$. □

2. On the other hand, if $\mu = 2$, we see that $L_2(x) = 2x(1 - x)$ on $[0, 1]$ is conjugate to $Q_0(x) = x^2$ on $[-1, 1]$. Recall in Exercises 1.1 the difference equation $x_{n+1} = 2x_n(1 - x_n)$ transforms to $y_{n+1} = y_n^2$ on setting $x_n = (1 - y_n)/2$. This is just the fact that L_2 and Q_0 are conjugate via $h(x) = -2x + 1$.

3. We can check that the logistic map L_4 is conjugate to $F : [-1, 1] \rightarrow [-1, 1]$, $F(x) = 2x^2 - 1$.

Exercises 6.3

1. Show that every quadratic polynomial $p(x) = ax^2 + bx + d$ is conjugate to a unique polynomial of the form $f_c(x) = x^2 + c$.

2. Prove that the logistic map L_4 is conjugate to $F : [-1, 1] \rightarrow [-1, 1]$, $F(x) = 2x^2 - 1$.

3. Show that the logistic map $L_\mu(x) = \mu x(1 - x)$, $x \in [0, 1]$ is conjugate to the logistic type map $F_\mu(x) = (2 - \mu)x(1 - x)$ ($\mu \neq 2$), via the linear conjugacy (which is defined on the interval with end points $\frac{1-\mu}{2-\mu}$ and $\frac{1}{2-\mu}$):

$$h(x) = \frac{\mu}{2 - \mu}x + \frac{1 - \mu}{2 - \mu}.$$

4. Let $f_c(x) = x^2 + c$. Show that if $c > -1/4$, there is a unique $\mu > 0$ such that f_c is conjugate to $L_\mu(x) = \mu x(1 - x)$ via a map of the form $h(x) = ax + b$.

5. (a) If $f_a(x) = ax$, $f_b(x) = bx$, $a, b \in \mathbb{R}$, defined on $\mathbb{R}$, when are they linearly conjugate?

(b) Show that $f_{1/2}$ and $f_{1/4}$ are conjugate via the map $h(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$.

6.4 Conjugacy and the Tent Family

We saw in Section 2.7 that for $\mu \geq (1 + \sqrt{5})/2$, the tent map T_μ has a point of period three, so by Sharkovsky's Theorem, it will have points of all possible periods. In this section we use a certain conjugacy to show that for $\mu > 1$, T_μ will have points of period 2^n for each $n \geq 1$. Our argument is based on that in [29]. We first show that the interval $[1/(1 + \mu), \mu/(1 + \mu)]$ is invariant under T_μ^2 when $1 < \mu \leq \sqrt{2}$.

The formula for T_μ^2 in Section 2.7 gives

$$T_\mu^2(x) = \begin{cases} \mu^2 x & \text{if } 0 \leq x \leq \frac{1}{2\mu} \\ \mu - \mu^2 x & \text{if } \frac{1}{2\mu} < x \leq \frac{1}{2} \\ \mu^2 x + \mu - \mu^2 & \text{if } \frac{1}{2} < x \leq 1 - \frac{1}{2\mu} \\ \mu^2 - \mu^2 x & \text{if } 1 - \frac{1}{2\mu} < x \leq 1 \end{cases}$$

Proposition 6.4.1 *For $1 < \mu \leq \sqrt{2}$, The restriction*

$$T_\mu^2 : \left[\frac{1}{1 + \mu}, \frac{\mu}{1 + \mu} \right] \rightarrow \left[\frac{1}{1 + \mu}, \frac{\mu}{1 + \mu} \right],$$

is well defined.

Proof. Note that for this range of values of μ , $1/(1 + \mu) < 1/2$ and $\mu/(1 + \mu) > 1/2$, so that $T_\mu(1/(1 + \mu)) = \mu/(1 + \mu)$ is an eventual fixed point since $T_\mu(\mu/(1 + \mu)) = \mu/(1 + \mu)$.

Let $x \in [1/(1 + \mu), \mu/(1 + \mu)]$, then from the formula for T_μ^2 and the fact that

$$\frac{1}{2\mu} < \frac{1}{1 + \mu} < \frac{\mu}{1 + \mu} < 1 - \frac{1}{2\mu},$$

we see that on this interval the minimum value of T_μ^2 occurs at $x = 1/2$ and this gives

$$T_\mu^2(x) \geq T_\mu^2(1/2) = \mu(1 - \mu/2) \geq \frac{1}{1 + \mu},$$

since this is equivalent to

$$\mu^3 - \mu^2 - 2\mu + 2 \leq 0, \quad \text{or} \quad (\mu - 1)(\mu^2 - 2) \leq 0,$$

where $1 < \mu \leq \sqrt{2}$.

Again assuming that $x \in [1/(1 + \mu), \mu/(1 + \mu)]$, we see that on the other hand, if $x \leq 1/2$, then $T_\mu(x) = \mu x > \mu/(1 + \mu) > 1/2$, so $T_\mu^2(x) = \mu(1 - \mu x) < \mu(1 - \mu/(1 + \mu)) = \mu/(1 + \mu)$, so $T_\mu^2(x) \in [1/(1 + \mu), \mu/(1 + \mu)]$.

If instead $x > 1/2$, then

$$T_\mu(x) = \mu(1 - x) > \mu(1 - \frac{\mu}{1 + \mu}) = \frac{\mu}{1 + \mu} > \frac{1}{2},$$

so

$$T_\mu^2(x) = \mu(1 - \mu(1 - x)) = \mu(1 - \mu + \mu x) \leq \mu(1 - \mu + \frac{\mu^2}{1 + \mu}) = \frac{\mu}{1 + \mu},$$

so again $T_\mu^2(x) \in [1/(1 + \mu), \mu/(1 + \mu)]$. $\square$

We use the above proposition to show that T_μ and $T_{\sqrt{\mu}}^2$ are conjugate when $T_{\sqrt{\mu}}^2$ is restricted to a suitable invariant subinterval.

Proposition 6.4.2 *For $1 < \mu \leq \sqrt{2}$, T_μ^2 restricted to the interval*

$$\left[\frac{1}{1 + \mu}, \frac{\mu}{1 + \mu} \right],$$

is conjugate to T_{μ^2} on $[0, 1]$.

Proof. From Proposition 6.4.1, we see that the given interval is invariant under T_μ^2 . Now we show that we actually have a linear conjugacy $h(x) = ax + b$:

$$h \circ T_\mu^2 = T_{\mu^2} \circ h,$$

where

$$h : \left[\frac{1}{1 + \mu}, \frac{\mu}{1 + \mu} \right] \rightarrow [0, 1],$$

and

$$a = \frac{1 + \mu}{1 - \mu}, \quad b = \frac{\mu}{\mu - 1},$$

where we can check that

$$h\left(\frac{\mu}{1 + \mu}\right) = 0, \quad \text{and} \quad h\left(\frac{1}{1 + \mu}\right) = 1.$$

If $0 \leq x \leq 1/2$, then since $h^{-1}(x) = x/a - b/a$, we can check that $1/2 \leq h^{-1}(x) \leq \mu/(1 + \mu) < 1 - 1/2\mu$, so that

$$\begin{aligned} h \circ T_\mu^2 \circ h^{-1}(x) &= h \circ T_\mu^2 \left(\frac{x}{a} - \frac{b}{a} \right) \\ &= h \left(\mu^2 \left(\frac{x}{a} - \frac{b}{a} \right) + \mu - \mu^2 \right) = \mu^2 x - \mu^2 b + a(\mu - \mu^2) + b = \mu^2 x = T_{\mu^2}(x). \end{aligned}$$

Similarly we can check that if $1/2 < x \leq 1$, then $1/2\mu < 1/(1+\mu) \leq h^{-1}(x) \leq 1/2$, so that

$$\begin{aligned} h \circ T_\mu^2 \circ h^{-1}(x) &= h \circ T_\mu^2\left(\frac{x}{a} - \frac{b}{a}\right) = h\left(\mu - \mu^2\left(\frac{x}{a} - \frac{b}{a}\right)\right) \\ &= a\mu - \mu^2x + \mu^2b + b = \mu^2(1-x) = T_{\mu^2}(x), \end{aligned}$$

i.e., in both cases we have $h \circ T_\mu^2 \circ h^{-1}(x) = T_{\mu^2}(x)$, giving the desired conjugacy. $\square$

Corollary 6.4.3 *For $1 < \mu \leq 2$, $T_{\sqrt{\mu}}^2$ restricted to the interval*

$$\left[\frac{\sqrt{\mu}-1}{\mu-1}, \frac{\mu-\sqrt{\mu}}{\mu-1} \right],$$

is conjugate to T_μ on $[0, 1]$.

We apply these results to give us information about the period points of T_μ :

Proof. Replace μ by $\sqrt{\mu}$ in the previous result. $\square$

Theorem 6.4.4 *For $1 < \mu \leq 2$, T_μ has a 2^n -cycle for each $n \in \mathbb{Z}^+$.*

Proof. We have seen that for each $\mu > 1$, T_μ has a period 2-point distinct from the fixed point of T_μ . In particular, as $\mu^2 > 1$, T_{μ^2} has a period 2-point distinct from the fixed point of T_{μ^2} . But by the last result, T_{μ^2} and T_μ^2 are conjugate, so T_μ^2 has a period 2-point distinct from the fixed point of T_μ^2 . This must be a period 4-point for T_μ , for if not it would be a period 2-point, giving a fixed point for T_μ^2 .

Continuing this argument, starting with a period 2-point for T_{μ^4} and the conjugacy between $T_{\mu^2}^2$ and T_{μ^4} we deduce that T_μ has a period 8-point. In this way, for each $n \in \mathbb{Z}^+$ we see that T_μ has a period 2^n -point. $\square$

Example 6.4.5 Consider the case where $\mu = 2$, then we see that T_2 , the standard tent map, is conjugate to $T_{\sqrt{2}}^2$ when it is restricted to the interval $[\sqrt{2}-1, 2-\sqrt{2}]$. This implies that $T_{\sqrt{2}}^2$ has the same dynamics as T_2 on this subinterval. For example, it must have a 3-cycle, say $\{c_1, c_2, c_3\}$, where the c_i 's are distinct and $T_{\sqrt{2}}^6(c_1) = c_1$. It follows that c_1 is a point of period 6 for $T_{\sqrt{2}}$, and in this way we deduce that $T_{\sqrt{2}}$ has $2k$ -cycles for each $k \in \mathbb{Z}^+$. We saw earlier that $\mu = (1 + \sqrt{5})/2$ is where period 3 is born for the tent family. In particular $T_{\sqrt{2}}$ has no 3-cycle, but if $\alpha = (1 + \sqrt{5})/2$ and using the fact that $T_{\sqrt{\alpha}}^2$ (suitably restricted) is conjugate to T_α , it follows that $T_{\sqrt{\alpha}}$ must have points of period 6.

Remark 6.4.6 1. Suppose that $\mu > 1$ and $\frac{\mu^2}{1+\mu^3} \leq \frac{1}{2}$, then $\frac{\mu^3}{1+\mu^3} = 1 - \frac{1}{1+\mu^3} \geq \frac{1}{2}$, so that

$$T_\mu\left(\frac{\mu}{1+\mu^3}\right) = \frac{\mu^2}{1+\mu^3}, \quad T_\mu\left(\frac{\mu^2}{1+\mu^3}\right) = \frac{\mu^3}{1+\mu^3} \quad \text{and} \quad T_\mu\left(\frac{\mu^3}{1+\mu^3}\right) = \frac{\mu}{1+\mu^3}.$$

We see that we have a 3-cycle:

$$\left\{ \frac{\mu}{1+\mu^3}, \frac{\mu^2}{1+\mu^3}, \frac{\mu^3}{1+\mu^3} \right\}.$$

This 3-cycle appears when $\mu > 1$ and $\frac{\mu^2}{1+\mu^3} \leq \frac{1}{2}$, or equivalently

$$\mu^3 - 2\mu^2 + 1 \geq 0, \quad \text{or} \quad (\mu - 1)(\mu^2 - \mu - 1) \geq 0.$$

We see that this will happen when $\mu \geq (1 + \sqrt{5})/2$. A similar analysis can be done for other periodic orbits. For example, if $\mu > 1$ and $\frac{\mu^3}{1+\mu^4} \leq \frac{1}{2}$ we get a 4-cycle, and this happens when $\mu^3 - \mu^2 - \mu - 1 \geq 0$.

2. Suppose that $1 < \mu < 2$, then if $x \in [\mu - \mu^2/2, \mu/2]$, we can check that $T_\mu(x) \in [\mu - \mu^2/2, \mu/2]$, so that this interval is an invariant set. For $1 < \mu < \sqrt{2}$, the smallest set invariant under T_μ is a collection of subintervals of $[\mu - \mu^2/2, \mu/2]$. If $\mu > \sqrt{2}$ this becomes all of the interval $[\mu - \mu^2/2, \mu/2]$, called the *Julia set* of T_μ (named after one of the early pioneers of chaotic dynamics, Gaston Julia, who worked on complex dynamics in particular in the early 1900's). For $\mu = 2$, the Julia set is all of $[0, 1]$. The bifurcation diagram for T_μ , $\mu > 1$ gives us some insight into what is happening here.

3. The conjugacy between T_2 and L_4 can be constructed by consideration of the periodic points of these maps. Since the period points are dense for each of these maps, by carefully ordering them according to their ordering in $[0, 1]$, we can define a map h by defining it on the periodic points. h is then defined on a dense subset of $[0, 1]$, into a dense subset. This map can be continuously extended to a homeomorphism of $[0, 1]$ with $h(0) = 0, h(1) = 1$. In this way it can be shown that the conjugation between T_2 and L_4 is unique. See the exercises for a proof that the conjugation between T_2 and L_4 is unique.

6.5 Renormalization

The previous example shows that T_μ^2 restricted to the interval

$$\left[\frac{\mu - 1}{\mu^2 - 1}, \frac{\mu^2 - \mu}{\mu^2 - 1} \right] = \left[\frac{1}{\mu + 1}, \frac{\mu}{\mu + 1} \right],$$

is conjugate to T_{μ^2} on $[0, 1]$ (where we are replacing μ by μ^2 in 6.4.2). How do we arrive at this conjugacy? Notice that for $\mu > 1$, T_μ has a fixed point $p_\mu = \mu/(\mu + 1)$ and another point $\hat{p}_\mu = 1/(\mu + 1)$ with $T_\mu(\hat{p}_\mu) = T_\mu(p_\mu)$, so it is eventually fixed. Let us look at the graph of T_μ^2 restricted to the interval $[\hat{p}_\mu, p_\mu]$. Inside the square shown we see that the graph obtained looks like an “upside-down” version of T_μ , and we consider the possibility that T_μ^2 restricted to the interval $[\hat{p}_\mu, p_\mu]$ is actually conjugate to T_μ (or in fact T_{μ^2}).

Define a linear map $h_\mu : [\hat{p}_\mu, p_\mu] \rightarrow [0, 1]$ of the form $h_\mu(x) = ax + b$ in such a way that $h_\mu(p_\mu) = 0$ and $h_\mu(\hat{p}_\mu) = 1$. We can check that

$$h_\mu(x) = \frac{1}{\hat{p}_\mu - p_\mu}(x - p_\mu), \quad \text{and} \quad h_\mu^{-1}(x) = (\hat{p}_\mu - p_\mu)x + p_\mu.$$

h_μ expands the interval $[\hat{p}_\mu, p_\mu]$ onto the interval $[0, 1]$ and changes the orientation. This is exactly the conjugacy defined in Proposition 6.4.2.

We define a *renormalization operator* of T_μ by

$$(RT_\mu)(x) = h_\mu \circ T_\mu^2 \circ h_\mu^{-1}(x).$$

What we actually showed in the previous section is that $(RT_\mu)(x) = T_{\mu^2}(x)$, giving us the conjugacy claimed. This procedure can be continued for T_μ^4 , T_μ^8 etc, and similar considerations can be made with the logistic map L_μ (see [20] for more details).

6.6 Conjugacy and Fundamental Domains

We have seen that two dynamical systems f and g with different dynamical properties cannot be conjugate. On the other hand, sometimes we have dynamical systems having seemingly very similar dynamical properties and which we would like to show are conjugate. This is sometimes possible using the notion of *fundamental domain*, a set on which we construct a map h in an arbitrary manner and show that it extends to a conjugacy on the whole space. We first illustrate this idea with homeomorphisms $f, g : \mathbb{R} \rightarrow \mathbb{R}$. We look at a fairly straightforward case where both homeomorphisms are order preserving and have no fixed points (in fact lie strictly above the line $y = x$).

Proposition 6.6.1 *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be homeomorphisms satisfying $f(x) > x$ and $g(x) > x$ for all $x \in \mathbb{R}$. Then f and g are conjugate.*

Proof. The idea for the proof is as follows: Select $x_0 \in \mathbb{R}$ arbitrarily and consider the 2-sided orbit

$$O_f(x_0) = \{f^n(x_0) : n \in \mathbb{Z}\} = \{\dots, x_{-1}, x_0, x_1, x_2, \dots\}.$$

Since $f(x) > x$ for all x , this is an increasing sequence: $\dots x_{-1} < x_0 < x_1 < x_2 < \dots$, so that the sets

$$\dots, [x_{-1}, x_0), [x_0, x_1), [x_1, x_2), \dots,$$

are disjoint and their union is all of $\mathbb{R}$. We must have $\lim_{n \rightarrow \infty} x_n = \infty$ since otherwise the limit would exist and would have to be a fixed point. There are no fixed points since $f(x) > x$ always.

The set $I = [x_0, f(x_0)) = [x_0, x_1)$ is called a *fundamental domain* for f . Set $J = [x_0, g(x_0))$ and define a map $h : I \rightarrow J$ arbitrarily as a continuous bijection (e.g., we can set $h(x_0) = x_0$ and $h(f(x_0)) = g(x_0)$ and then linearly from I to J).

Now every other orbit of f intertwines with $O_f(x_0)$: if $y_0 \in (x_0, x_1)$, then $y_n = f^n(y_0) \in f^n(I)$, so lies between x_i and x_{i+1} . It follows that every orbit has a unique member in the interval $[x_0, x_1)$ and we use this to extend the definition of h to all of $\mathbb{R}$.

If $x \in f^n(I)$ we define $h(x)$ by mapping x back to I via f^{-n} , then using $h(f^{-n}(x))$ which is well defined, and then mapping back to $g^n(J)$ using g^n . I.e., if $x \in f^n(I)$, $n \in \mathbb{Z}$, define

$$h(x) = g^n \circ h \circ f^{-n}(x).$$

In this way, h is defined on all of $\mathbb{R}$. We can check that h is one-to-one. It is onto because $h(f^n(I)) = g^n(J)$ for each n , and we can check that it is continuous. Finally, because of the definition of h , if $x \in \mathbb{R}$, then $x \in f^n(I)$ for some $n \in \mathbb{Z}$, so $x = f^n(y)$ for some $y \in I$. Then

$$g \circ h(x) = g(g^n \circ h \circ f^{-n}(x)) = g^{n+1} \circ h \circ f^{-(n+1)}(f(x)) = h \circ f(x),$$

so that f and g are conjugate. □

Examples 6.6.2 1. The above argument can be generalized to the situation where $f(x)$ and $g(x)$ are homeomorphisms with corresponding fixed points. To be more precise, consider a homeomorphism $f : [0, 1] \rightarrow [0, 1]$ which is orientation preserving, so that $f(0) = 0$, $f(1) = 1$ and f is increasing. Suppose that f has fixed points (in addition to 0 and 1), at $c_1, c_2, \dots, c_n$, then f^2 has the same collection of fixed points (no additional fixed points as f cannot have points of period 2 or higher). If $f(x) > x$ for $c_k < x < c_{k+1}$, then we can use the argument of the proposition to construct a homeomorphism between f and f^2 , and do the same for each interval

$[c_i, c_{i+1}]$ (treating the case where $f(x) < x$ in an analogous way). In this way we see that f and f^2 are conjugate maps.

2. Consider the logistic maps $L_\mu(x) = \mu x(1 - x)$ for various values of $\mu \in (0, 4]$ and $x \in [0, 1]$. We first show that for $0 < \mu < \lambda \leq 1$, L_μ and L_λ are conjugate. There is a slight complication here as these maps are not increasing, but they do have an attracting fixed point at 0, and we saw earlier that the basin of attraction is all of $[0, 1]$. We first deal with the interval on which the maps are increasing, $[0, 1/2]$, and look at the restriction of the functions to this interval.

Our aim is to construct a homeomorphism $h : [0, 1] \rightarrow [0, 1]$ with the property $L_\lambda \circ h = h \circ L_\mu$. Take $(L_\mu(1/2), 1/2] = (\mu/4, 1/2]$ as a fundamental domain for L_μ and $(L_\lambda(1/2), 1/2] = (\lambda/4, 1/2]$ as a fundamental domain for L_λ . Define $h : (\mu/4, 1/2] \rightarrow (\lambda/4, 1/2]$ by $h(1/2) = 1/2$ and $h(\mu/4) = \lambda/4$ and then linearly on the remainder of the interval.

Set $I = (\mu/4, 1/2]$ and $J = (\lambda/4, 1/2]$, then since 0 is an attracting fixed point, the intervals $L_\mu^n(I)$ and $L_\lambda^n(J)$ are disjoint for $n \in \mathbb{Z}^+$, and their union is all of $(0, 1/2]$. Extend the definition of h so that it is defined on $(0, 1/2]$ by;

$$h(x) = L_\lambda^n \circ h \circ L_\mu^{-n}(x), \quad \text{for } x \in L_\mu^n(I).$$

We can now check that h is continuous and increasing on $[0, 1/2]$ when we set $h(0) = 0$.

Now define h on $(1/2, 1]$ by setting $h(1 - x) = 1 - h(x)$ for $x \in [0, 1/2]$, clearly giving a homeomorphism on $[0, 1]$. Then

$$L_\lambda(h(1 - x)) = L_\lambda(1 - h(x)) = L_\lambda(h(x)) = h(L_\mu(x)) = h(L_\mu(1 - x)),$$

so that h is the required conjugation. $\square$

3. A similar proof shows that L_μ and L_λ are conjugate whenever $1 < \mu < \lambda < 2$. Look at the intervals $[0, 1 - 1/\mu]$ and $[1 - 1/\mu, 1/2]$ separately and the fact that $1 - 1/\mu$ is an attracting fixed point, then use the symmetry about the point $x = 1/2$.

However, these maps cannot be conjugate to L_2 since any conjugating map $h : [0, 1] \rightarrow [0, 1]$ must have the property that $h(1/2) = 1/2$ (see the exercises). This leads to a contradiction.

4. The maps L_4 and L_μ , $\mu \in (0, 4)$ cannot be conjugate since $L_4 : [0, 1] \rightarrow [0, 1]$ is an onto map, but L_μ is not (see the exercises).

Exercises 6.6

1. (a) Let $a, b \in (0, 1)$ and $f_a(x) = ax$, $f_b(x) = bx$ be dynamical systems on $[0, 1]$. We saw in Exercises 6.3 that these maps need not be linearly conjugate. Prove that f_a and f_b are conjugate (Hint: Use the method of examples in this section).

(b) Let $g : [0, 1] \rightarrow [0, 1]$ be continuous, strictly increasing with $g(0) = 0$ and $g(x) < x$ for all $x \in (0, 1]$. Prove that g is conjugate to f_a for any $a \in (0, 1)$.

2. Consider the tent map $T_{\sqrt{2}}$.

(i) Show that $x = 1/2$ is an eventual fixed point for $T_{\sqrt{2}}$.

(ii) Use Section 6.4 to show that there is a subinterval of $[0, 1]$ on which $T_{\sqrt{2}}$ is conjugate to T_2 .

(iii) Deduce that $T_{\sqrt{2}}$ has periodic points of period $2k$ for any $k > 1$, but no points of odd period greater than 1.

(iv) Prove that if $\mu > \sqrt{2}$, then T_μ has a 3-cycle.

(v) Prove that there is an interval on which $T_{\sqrt{2}}$ is chaotic.

3. Let $0 < \lambda, \mu < 1$. If $h : [0, 1] \rightarrow [0, 1]$ is an orientation preserving homeomorphism with $h \circ L_\mu(x) = L_\lambda \circ h(x)$ for all $x \in [0, 1]$, show that $h(1/2) = 1/2$ (Hint: h is a conjugation between two different logistic maps with $h \circ L_\mu(x) = L_\lambda \circ h(x)$. Note that this equation also holds if we replace x by $1 - x$. Use this to deduce that $h(x) + h(1 - x) = 1$ for all $x \in [0, 1]$).

4. Use exercise 2 above to deduce that $h(\mu/4) = \lambda/4$. It follows that the orientation preserving homeomorphism of Example 6.6.2 can be extended in only one way from $[\mu/4, 1/2]$ to $[\lambda/4, 1/2]$.

5*. Prove that for the logistic map L_μ , if $0 < \mu \leq 2$, then L_μ is conjugate to L_μ^2 , the composition of L_μ with itself. Show that this is not true for $\mu > 2$. What is the corresponding result for the tent family?

6. Let $S_\mu(x) = \mu \sin(x)$. Prove that if $0 < \mu < \lambda < 1$, then S_μ and S_λ are conjugate.

7. Prove that the rotation $R_a : S^1 \rightarrow S^1$, $R_a(z) = az$ is conjugate to the map $T_\alpha : [0, 1) \rightarrow [0, 1)$, $T_\alpha(x) = x + \alpha \pmod{1}$, when $a = e^{2\pi i \alpha}$. Can R_a be conjugate to R_b for $a \neq b$?

8. Prove that $T_\alpha : [0, 1) \rightarrow [0, 1)$, $T_\alpha(x) = x + \alpha \pmod{1}$ is conjugate to its inverse map $T_\alpha^{-1}(x) = x - \alpha \pmod{1}$. Can T_α be conjugate to T_α^2 ?

9. The aim of this exercise is to show the uniqueness of the conjugacy between the tent map T_2 and the logistic map L_4 .

(i) Check that this conjugacy $k : [0, 1] \rightarrow [0, 1]$ is given by,

$$k(x) = \frac{2}{\pi} \arcsin(\sqrt{x}); \quad T_2 \circ k = k \circ L_4.$$

(ii) Suppose that $h : [0, 1] \rightarrow [0, 1]$ is another conjugacy between T_2 and L_4 , then $h(0) = 0$, $h(1) = 1$ and h is a strictly increasing continuous function (why?), i.e., h is an orientation preserving homeomorphism of $[0, 1]$.

(iii) Show that h maps the local maxima (respectively minima) of T_2^n to local maxima (respectively minima) of L_4^n .

(iv) Use the fact that any such conjugation is order preserving to show that $h(x) = k(x)$ at all local maxima and local minima.

(v) Use the continuity of h and k to deduce that $h(x) = k(x)$ for all $x \in [0, 1]$.

(vi) Deduce that there is no C^1 conjugacy between T_2 and L_4 .

10. Use the above exercise to show that if $L_4(x) = 4x(1 - x)$, then L_4^n has turning points at $\sin^2(k\pi/2^{n+1})$, for $k = 1, 2, \dots, 2^{n+1} - 1$.

11. Use the fact that the conjugation between T_2 and L_4 is unique to show that if $\phi : [0, 1] \rightarrow [0, 1]$ is a homeomorphism satisfying $L_4 \circ \phi = \phi \circ L_4$, then $\phi(x) = x$ for all $x \in [0, 1]$, i.e., ϕ is the identity map (hint: first show that $k \circ \phi$ is also a conjugation between T_2 and L_4).

12*. Let T_μ be the tent map. Show that if $\mu > \sqrt{2}$, then for each open interval $U \subset [0, 1]$, there exists $n > 0$ such that

$$[T_\mu^2(1/2), T_\mu(1/2)] \subseteq T_\mu^n(U).$$

(Hint: Use the fact that $|T_\mu(U)| \geq \mu|U|$ if U does not contain $1/2$, so that the length keeps increasing. We claim there exists $m > 0$ such that $T^m(U)$ and $T^{m+1}(U)$ both contain $1/2$, for if not, $|T^{m+2}(U)| \geq \mu^2|U|/2$ for all $m \in \mathbb{Z}^+$, a contradiction, since this eventually exceeds 1).

Chapter 7. Singer's Theorem

We shall now show that the logistic map $L_\mu : [0, 1] \rightarrow [0, 1]$, $L_\mu(x) = \mu x(1 - x)$ with $0 < \mu < 4$ has at most one attracting cycle. We use a result due to Singer (1978) which is applicable to maps having a negative Schwarzian derivative. Recall that a map $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^3 if $f'''(x)$ exists and is continuous.

7.1 The Schwarzian Derivative Revisited

Recall the Schwarzian derivative of $f(x)$ is:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2 = F'(x) - \frac{1}{2} [F(x)]^2,$$

where $F(x) = \frac{f''(x)}{f'(x)}$.

Our first goal is to show that many polynomials have negative Schwarzian derivatives.

Lemma 7.1.1 *Let $f(x)$ be a polynomial of degree n for which all the roots of its derivative $f'(x)$ are distinct and real. Then $Sf(x) < 0$ for all x .*

Proof. Suppose that the derivative of $f(x)$ is given by

$$f'(x) = a(x - r_1)(x - r_2) \cdots (x - r_{n-1}),$$

where $a \in \mathbb{R}$, then

$$F(x) = \frac{f''(x)}{f'(x)} = (\ln f'(x))' = \sum_{i=1}^{n-1} \frac{1}{x - r_i},$$

and so

$$F'(x) = - \sum_{i=1}^{n-1} \frac{1}{(x - r_i)^2}.$$

Now put this into the Schwarzian derivative:

$$Sf(x) = F'(x) - \frac{1}{2} [F(x)]^2 < 0.$$

□

Lemma 7.1.2 *Assume that f is a C^3 map on $\mathbb{R}$, then*

(i) $S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x)$.

(ii) *If $Sf < 0$ and $Sg < 0$, then $S(f \circ g) < 0$.*

(iii) If $Sf < 0$, then $Sf^k < 0$ for all $k \in \mathbb{Z}^+$.

Proof. (i) As above we have $F(x) = \frac{f''(x)}{f'(x)}$, so set $G(x) = \frac{g''(x)}{g'(x)}$ and $H(x) = \frac{h''(x)}{h'(x)}$, where $h = f \circ g$. Then

$$h'(x) = f'(g(x)) \cdot g'(x), \quad h''(x) = (f''(g(x))g'(x)^2 + (f'(g(x)))g''(x)),$$

so that

$$\begin{aligned} H(x) &= \frac{f''(g(x))g'(x)^2 + f'(g(x))g''(x)}{f'(g(x))g'(x)} \\ &= \frac{f''(g(x))g'(x)}{f'(g(x))} + \frac{g''(x)}{g'(x)} = F(g(x))g'(x) + G(x). \end{aligned}$$

This gives

$$H'(x) = F'(g(x))g'(x)^2 + F(g(x))g''(x) + G'(x),$$

so that

$$\begin{aligned} S(f \circ g)(x) &= H'(x) - \frac{1}{2}H^2(x) \\ &= \left[F'(g(x)) - \frac{1}{2}F^2(g(x)) \right] g'(x)^2 + F(g(x))g''(x) - F(g(x))g'(x)G(x) + G'(x) - \frac{1}{2}G^2(x) \\ &= Sf(g(x)) \cdot g'(x)^2 + Sg(x), \end{aligned}$$

since $G(x) = g''(x)/g'(x)$.

(ii) is now immediate, and (iii) follows using induction. $\square$

Example 7.1.3 Let $g(x) = \frac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$, a linear fractional transformation. A direct calculation shows that $Sg(x) = 0$ everywhere in its domain. It now follows from Lemma 1 that if $h(x) = g(f(x)) = \frac{af(x)+b}{cf(x)+d}$, then $Sh(x) = Sf(x)$.

We now prove a version of Singer's Theorem. Recall that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $c \in \mathbb{R}$ is an attracting fixed point or attracting periodic point, then the basin of attraction $B_f(c)$ is an open set. Denote by W the maximal open interval contained in $B_f(c)$ which contains c (called the *immediate basin of attraction of c*).

Theorem 7.1.4 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^3 map with negative Schwarzian derivative. If c is an attracting periodic point for f , then either:

(i) the immediate basin of attraction of c extends to ∞ or $-\infty$, or

(ii) *there is a critical point of f (i.e., a root of $f'(x) = 0$), whose orbit is attracted to the orbit of c under f .*

Proof. We first look at the case where c is a fixed point of f . Suppose that its immediate basin of attraction is the open interval W and that (i) does not hold, then W is a bounded set.

Thus $W = (a, b)$ for some $a, b \in \mathbb{R}$. Because of the continuity of f and the fact that $f(a), f(b) \notin (a, b)$, there are three possibilities for $f(a)$ and $f(b)$.

Case 1: $f(a) = f(b)$ - this will happen for example if a is a fixed point and b is an eventual fixed point. It follows from the Mean Value Theorem that (a, b) contains a critical point of f .

Case 2: $f(a) = a$ and $f(b) = b$, then by the Mean Value Theorem, there are points $x_1 \in (a, c)$ and $x_2 \in (c, b)$ such that $f'(x_1) = f'(x_2) = 1$ (see picture). But since c is an attracting fixed point, $|f'(c)| \leq 1$. It follows that either $f'(x_0) = 0$ for some $x_0 \in (x_1, x_2)$, or $f'(x)$ has a minimum value $f'(x_0) > 0$. In the latter case we have $f'(x_0) > 0$, $f''(x_0) = 0$ and $f'''(x_0) > 0$, so that $Sf(x_0) > 0$, contradicting the Schwarzian derivative being everywhere negative.

Case 3: $f(a) = b$ and $f(b) = a$. Here c is fixed by f^2 and so are a and b , so that $(f^2)'$ has a zero x_0 in (a, b) (as in Case 1). But

$$(f^2)'(x_0) = (f'(f(x_0)))f'(x_0) = 0,$$

so either x_0 or $f(x_0)$ is a root of f' , but both lie in (a, b) .

Now suppose that c is a point of period k , then $f^k(c) = c$, an attracting fixed point for f^k , then from our earlier arguments, the immediate basin of attraction of c (for f^k) contains a critical point of f^k , say x_0 :

$$(f^k)'(x_0) = f'((x_0)f'(f(x_0)) \cdots f'(f^{k-1}(x_0))) = 0,$$

so that $f'(f^m(x_0)) = 0$ for some $0 \leq m < k$. In this case $f^m(x_0) \in f^m(W) \subset W$, the basin of attraction of c . $\square$

Example 7.1.5 Consider the map $f(x) = x - x^5$. We see that $f'(x)$ has two real roots: $\pm(1/5)^{1/4}$, and $f''(x)$ and $f'''(x)$ are both continuous. We have $f''(x) = -20x^3$, and $f'''(x) = -60x^2$.

Substituting these into the Schwarzian derivative gives:

$$Sf(x) = (-60x^2)/(1 - 5x^4) - 3/2[(-20x^3)/(1 - 5x^4)]^2 = \frac{-60x^2}{(1 - 5x^4)^2}(1 + 5x^4),$$

and this is always negative. We can check that the critical points are in the basin of attraction of the fixed point $x = 0$.

Example 7.1.6 The map $f(x) = 3x/4 + x^3$ cannot have a negative Schwarzian derivative. It has fixed points $x = 0$ and $x = \pm 1/2$, 0 being attracting with bounded basin of attraction, but f has no critical points.

7.2 Singer's Theorem

Corollary 7.2.1 *Let $f : [0, 1] \rightarrow [0, 1]$ be a C^3 map with $Sf(x) < 0$ for all x . The basin of attraction of an attracting cycle contains 0, 1 or a critical point of $f(x)$.*

Proof. If $J = (a, b)$, $0 < a < b < 1$, is the basin of attraction of an attracting cycle, then we have seen above that it must contain a critical point of f . Any other attracting cycles will be of the form $[0, a)$ or $(b, 1]$, so will contain 0 or 1. $\square$

Example 7.2.2 We now see that the logistic map $L_\mu(x) = \mu x(1 - x)$, $0 < \mu < 4$, $x \in [0, 1]$, has at most one attracting periodic cycle. If $0 < \mu \leq 1$, 0 is the only attracting fixed point, having basin of attractions $[0, 1]$. For $1 < \mu < 4$, L_μ has exactly one critical point $x_0 = 1/2$.

Since $L'_\mu(0) = \mu > 1$, the fixed point 0 is unstable; therefore $[0, a)$ cannot be a basin of attraction. Furthermore, $L_\mu(1) = 0$ and hence $(b, 1]$ is not a basin of attraction either. Since $SL_\mu(x) < 0$ everywhere (at $x = 1/2$, $\lim_{x \rightarrow 1/2} L_\mu(x) = -\infty$), we conclude that there is at most one attracting periodic cycle in $(0, 1)$ and the result follows.

If we look at the bifurcation diagram of the logistic map, it follows, for example that where we see six horizontal lines, we have an attracting 6-cycle, and not two attracting 3-cycles.

Remark 7.2.3 In a similar way we get Singers Theorem [59], proved by David Singer in 1978. This theorem is actually a real version of a theorem about complex polynomials proved by the French mathematician Gaston Julia in 1918 [32]:

Suppose that f is a C^3 map on a closed interval I such that $Sf(x) < 0$, for all $x \in I$. If f has n critical points in I , then f has at most $n + 2$ attracting cycles.

Example 7.2.4 Let $G(x) = \lambda \arctan(x)$, $\lambda \neq 0$. Then $G'(x) = \lambda/(1+x^2)$. Clearly $G(x)$ has no critical points. Now, if $|\lambda| < 1$, then $c = 0$ is an asymptotically stable fixed point with basin of attraction $(-\infty, \infty)$. If $\lambda > 1$, then G has two attracting fixed points x_1 and x_2 with basins of attraction $(-\infty, 0)$ and $(0, \infty)$, respectively. Finally, if $\lambda < -1$, then G has an attracting 2-cycle $\{\bar{x}_1, \bar{x}_2\}$ with basin of attraction $(-\infty, 0) \cup (0, \infty)$.

Exercises 7.2

1. If $G(x) = \lambda \arctan(x)$, ($\lambda \neq 0$), show that the Schwarzian derivative is

$$SG(x) = \frac{-2}{(1+x^2)^2}.$$

Use this to verify the conclusion from Example 7.2.4.

2. If $f(x) = \frac{ax+b}{cx+d}$, we have seen that the Schwarzian derivative satisfies $Sf(x) = 0$. Now suppose that f is a function for which $Sf(x) = 0$. Show

(i) $(f''(x))^2/(f'(x))^3 = \text{constant}$,

- (ii) $f(x)$ is of the form $f(x) = \frac{ax+b}{cx+d}$ (Hint: Set $y = f'$ to obtain a separable differential equation of the form $y' = cy^{3/2}$).