

Chapter 8. Fractals

Many objects in nature have been modeled mathematically by approximating the curves or surfaces involved by 'smooth' curves or surfaces, in order that the calculus be applied to their study. However, in recent years it has been realized that for certain of these objects, the calculus is not the best tool for their study.

For example, the motion of a particle suspended in a fluid (Brownian motion), the length of the coastline of an island or the surface area of the human lung. The length of a coast line is dependent on how carefully it is measured; for such an object the more closely you look at it, the more irregular it appears and the greater the length appears to be. The study of such objects has resulted in a new area of mathematics called Fractal Geometry. Fractal geometry was popularized by the mathematician Benoit Mandelbrot and it was him who coined the term *fractal* in 1977 ([42]). Mandelbrot was originally from Poland, educated in France and later moved to the United States is particularly famous for the so called "Mandelbrot set".

Much of the current interest in fractals is a consequence of Mandelbrot's work. His computer simulations of maps of the complex plane have resulted in extremely complicated and beautiful fractals. The mathematical work was initiated by Cayley, Fatou and Julia in the late 19th and early 20th centuries, but progress ceased until the development of the electronic computer. Later we shall see how fractals arise from the study of complex analytic maps.

8.1 Examples of Fractals

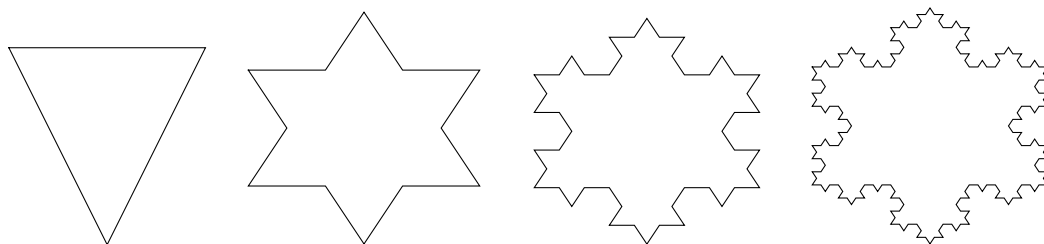
Probably the first published example of a fractal was given by Karl Weierstrass in 1872. He constructed an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous everywhere, but nowhere differentiable. Until this time mathematicians believed that continuous functions had to be differentiable at "most" points, and this was often tacitly assumed. We saw earlier how to construct the Cantor set (published in 1883), probably the simplest example of a fractal.

Another early example is due to Helge von Koch (1904), who constructed the famous "Koch Snowflake". Starting with an equilateral triangle of side length 1 unit, three equilateral triangles are constructed (one on each side as shown), each having side length $1/3$. This construction is then continued so that at each stage, an equilateral triangle of side length $1/3$ that of the previous ones is added to each exposed line. The Koch Snowflake itself is a limiting curve in a sense to be described shortly.

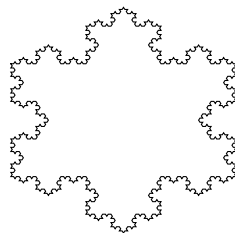
The Koch Snowflake curve is constructed inductively as follows:

- (i) Start with an equilateral triangle with side length = 1, the perimeter of the resulting curve is $L(0) = 3$.
- (ii) Now add equilateral triangles of side length $1/3$. The perimeter is now $L(1) = 3 \times 4 \times 1/3 = 4$.
- (iii) Add new equilateral triangles as seen below by taking $n = 2$. The perimeter is now $L(2) = 3 \times 4^2 \times (1/3^2) = 4^2/3$.
- (iv) Continue in this way so that at the n th stage the perimeter is $L(n) = 4^n/3^{n-1}$.

Take in turn $n = 3, 4, \dots$ to get a good idea of what the limiting curve, (the Koch snowflake) looks like. As $n \rightarrow \infty$, $L(n) \rightarrow \infty$, so that the Koch Snowflake is a curve having infinite length. It can be seen to be continuous, and it can be shown to be nowhere differentiable. The area enclosed by the curve is easily found and is clearly finite.



The first 4 iterations of the Koch Snowflake.



The Koch Snowflake

The Koch snowflake is an example of a fractal in the sense that it has the self-similarity property typical of fractals - as we zoom in on any part of the curve (no matter how far), it resembles (gives an exact copy) of what we saw previously. It is a fractal curve in $\mathbb{R}^2$, which typically are continuous curves, differentiable nowhere and consequently have infinite length.

8.2 An Intuitive Introduction to the Idea of Fractal Dimension

The idea of (topological) dimension is intuitively clear. For example, a point or a number of points has 0-dimensions, a curve is 1-dimensional, a surface 2-dimensional, etc. Thus the topological dimension of the snowflake curve is 1. More precisely:

Definition 8.2.1 A set $K \subseteq \mathbb{R}^n$ has *topological dimension* 0 if for every point $x \in K$ there is an open ball $B_\delta(x)$ in $\mathbb{R}^n$ having arbitrarily small radius, whose boundary does not intersect K .

K has topological dimension $k \in \mathbb{Z}^+$ if every point $x \in K$ is surrounded by an open ball $B_\delta(x)$ having arbitrarily small radius whose boundary intersects K in a set of topological dimension $k - 1$, and k is the least positive integer with this property.

Any discrete set such as the Cantor set or the set of rationals will have topological dimension 0. The topological dimension of a line, a circle or the Koch curve is 1. A filled in circle or square will have topological dimension 2 and for a solid cube or sphere it will be 3.

It turns out that for sets such as the snowflake curve and the Cantor set, there is another very useful idea of dimension, originally called *Hausdorff-Besicovitch dimension*, now often called *fractal dimension* (following B. Mandelbrot). We motivate its definition as follows:

- (i) Given a piece of string, two copies of it result in a string “twice the size”.
- (ii) For a square we need 4 copies.
- (iii) For a cube we need 8 copies.
- (iv) For a 4-dimensional cube we need 16 copies. i.e. to double the (side length) of a d -dimensional cube we need $c = 2^d$ copies, so that $d = (\log c)/(\log 2)$,

Thus for example, if we have an object whose size doubles if three copies are stuck together, then it would have fractal dimension $\log 3 / \log 2$.

Returning to the snowflake curve, notice that (one side of it) is made up of 4 copies of itself; each $1/3$ of the size, so $a = 3$, $c = 4$ and

$$d = \frac{\log c}{\log a} = \frac{\log 4}{\log 3} = 1.2616 \dots$$

Definition 8.2.2 This number d is defined to be the *fractal dimension* of the curve.

The Cantor set is seen to be made up of two copies of itself, each reduced in size by a factor of three, so that $c = 2$ and $a = 3$ and $d = \log 2 / \log 3$, again a fractal since the topological dimension of C is zero.

Roughly speaking a fractal is defined to be a geometrical object whose fractal dimension is strictly greater than its (topological) dimension (this was the original definition due to Mandelbrot). The coastline of Britain has $d = 1.25$ (approximately), so it is very rough. That of the U.S. is closer to 1 (fairly smooth).

Generally, fractals have the property of being self-similar, i.e. they do not change their appearance significantly when viewed under a microscope of arbitrary magnifying power. This self similarity can take a linear form as in the snow flake curve, or a non-linear form as in the Mandelbrot set (see later).

8.3 Box Counting Dimension

We try to make the discussion of the last section more precise. A *box* in $\mathbb{R}^n$ is a set of the form

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \{(x_1, x_2, \dots, x_n) : a_i \leq x_i \leq b_i, \ i = 1, 2, \dots, n\}.$$

Let $K \subseteq \mathbb{R}^n$ and set $N_\delta(K)$ = the smallest number of boxes of equal side length δ needed to cover the set K . As $\delta \rightarrow 0$, we need more such boxes, so that $N_\delta(K)$ will increase. The idea is that as δ decreases, the number of sets required to cover K increases as some power of the length of the boxes:

$$N_\delta \sim \delta^{-d}.$$

This power d is the *box counting dimension*, which is defined more formally as follows:

Definition 8.3.1 The *box counting dimension* of $K \subseteq \mathbb{R}^n$ is given by

$$\dim(K) = \lim_{\delta \rightarrow 0} \frac{\log N_\delta(K)}{-\log \delta},$$

if this limit exists (it is independent of the base of the logarithm).

Examples 8.3.2 1. Let $K = [0, 1]$, the unit interval in $\mathbb{R}$. We can cover K with one interval of length $\delta = 1$, two intervals of length $\delta = 1/2$. In general we need 2^n intervals of length $\delta = 2^{-n}$ so that

$$\dim([0, 1]) = \lim_{n \rightarrow \infty} \frac{\log 2^n}{-\log 2^{-n}} = 1,$$

which is what we would expect (strictly speaking we have only shown that $\dim([0, 1]) \leq 1$ since we have not shown that the covering gives the minimum N).

It can be shown that if $K \subseteq \mathbb{R}^n$ contains an open subset of $\mathbb{R}^n$, then $\dim(K) = n$ (what is called the *topological dimension*).

2. The Cantor set C has $N_1(C) = 1$ (using the single interval $[0, 1]$ to cover C). Dividing $[0, 1]$ into three equal subintervals, we can cover C with two intervals of length $1/3$.

Repeating this process, we can cover C with 2^n subinterval of length 3^{-n} , so that $N_{3^{-n}}(C) = 2^n$ and

$$\dim(C) = \lim_{n \rightarrow \infty} \frac{\log 2^n}{-\log 3^{-n}} = \frac{\log 2}{\log 3}.$$

3. The Menger sponge M can be shown to have box counting dimension equal to $\dim(M) = \lim_{n \rightarrow \infty} \frac{\log 20^n}{\log 3^{-n}} = \frac{\log 20}{\log 3} \sim 2.73$, since when $\delta = 1$, $N_1(M) = 1$, when $\delta = 1/3$, $N_\delta(M) = 20$ and continuing in this way gives the result.

8.4. The Mathematical Theory of Fractals

In the remainder of this chapter we attempt to give a rigorous definition of the idea of a limiting set. For example, the Koch Snowflake is the limiting curve of a sequence of curves in $\mathbb{R}^2$. We intend to make rigorous this notion. In order to do so, we need to define a metric on “sets of sets”, known as the *Hausdorff metric*, and then we can look at the convergence of sequences in the resulting metric spaces.

8.4.1 Contraction Mappings and Complete Metric Spaces

Given a sequence x_n in a metric space X , it is possible that the sequence converges, but not to a point of X . For example, consider the following sequence in $\mathbb{Q}$ (the set of rationals with its usual metric $d(x, y) = |x - y|$ for $x, y \in \mathbb{Q}$):

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.14$, $x_3 = 1.141$, and x_n being the member of $\mathbb{Q}$ equal to $1.141\dots$, with a decimal expansion consisting of 1 followed by the first n terms in the decimal expansion of $\sqrt{2}$, followed by 0's. It is clear that x_n is a sequence in $\mathbb{Q}$, but $\lim_{n \rightarrow \infty} x_n = \sqrt{2} \notin \mathbb{Q}$. The reals $\mathbb{R}$ and complex numbers $\mathbb{C}$ do not have this type of difficulty, they are what we call *complete metric spaces*.

Definition 8.4.2 Let $\{x_n\}$ be a sequence in a metric space (X, d) , then $\{x_n\}$ is a *Cauchy sequence* if given any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ for which

$$m, n > N \Rightarrow d(x_n, x_m) < \epsilon.$$

Definition 8.4.3 A metric space (X, d) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to a member of X .

Proposition 8.4.4 In a metric space (X, d) , any convergent sequence is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a convergent sequence, converging to $x \in X$, then given $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$n > N \Rightarrow d(x, x_n) < \epsilon/2.$$

By the triangle inequality, if $m, n > N$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon,$$

so that $\{x_n\}$ is a Cauchy sequence. $\square$

Example 8.4.5 The sequence $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, is not a Cauchy sequence in $\mathbb{R}$.

Proof. A sequence $\{x_n\}$ in a metric space (X, d) fails to be Cauchy if we can find $\epsilon > 0$ such that for all $N \in \mathbb{Z}^+$, we can find x_n, x_m with $n, m \geq N$ and $d(x_n, x_m) \geq \epsilon$. We shall see that it suffices to take $\epsilon = 1/2$. In this case, $d(x_n, x_m) = |x_n - x_m|$.

Set $n = 2N$ and $m = N$, then

$$|x_{2N} - x_N| = \frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} \cdots + \frac{1}{2N} > \frac{N}{2N} = \frac{1}{2},$$

so that x_n is not a Cauchy sequence. $\square$

The problem is that the converse of Proposition 8.4.4 is not true. It can be seen that the rational numbers do not constitute a complete metric space since for example, the sequence mentioned above, where x_n is the first n terms of the decimal expansion of $\sqrt{2}$, followed by 0's, can be seen to be a Cauchy sequence in $\mathbb{Q}$ (with respect to the usual metric), but does not converge in $\mathbb{Q}$. The reals $\mathbb{R}$, $\mathbb{R}^n$ and the complex numbers $\mathbb{C}$ are examples of complete metric spaces. The completeness of $\mathbb{R}$ is a consequence of the *Completeness Axiom* (See the Appendix), which says that any non-empty, bounded subset of $\mathbb{R}$ has a *least upper bound* (called the *supremum*), and the completeness of $\mathbb{R}^n$ and $\mathbb{C}$ can be deduced from this.

8.5 The Contraction Mapping Theorem and Self-Similar Sets

The main property of complete metric spaces we shall study is concerned with functions $f : X \rightarrow X$ having the effect of bringing points closer together.

Definition 8.5.1 A function $f : X \rightarrow X$ of the metric space (X, d) is called a *contraction mapping* if there exists a real number α , $0 < \alpha < 1$ satisfying

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y), \quad \text{for all } x, y \in X.$$

The constant α is called the *contraction constant* of f .

We shall see that contraction mappings always have a fixed point. The following theorem is called the Contraction Mapping Theorem, or the Banach Fixed-Point Theorem, after the Polish mathematician Stefan Banach who stated and proved the theorem in 1922. The proof we shall give is due to R. Palais [48].

Theorem 8.5.2 *Let $f : X \rightarrow X$ be a contraction mapping on a non-empty complete metric space X , then f has a unique fixed point. In addition, if p is the fixed point, then $p = \lim_{n \rightarrow \infty} f^n(x)$ for every $x \in X$.*

Proof. Choose $x_0 \in X$ arbitrarily and set $x_n = f^n(x_0)$, then we shall show that $\{x_n\}$ is a Cauchy sequence and hence is convergent. Suppose that $0 < \alpha < 1$ has the property that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$ for $x, y \in X$. Then inductively we see that for any $k \in \mathbb{Z}^+$

$$d(f^k(x), f^k(y)) \leq \alpha^k \cdot d(x, y).$$

By the triangle inequality we have for $x, y \in X$:

$$\begin{aligned} d(x, y) &\leq d(x, f(x)) + d(f(x), f(y)) + d(f(y), y) \\ &\leq d(x, f(x)) + \alpha d(x, y) + d(f(y), y). \end{aligned}$$

Solving for $d(x, y)$ gives the “Fundamental Contraction Inequality”:

$$d(x, y) \leq \frac{d(f(x), x) + d(f(y), y)}{1 - \alpha}.$$

Note that if x and y are both fixed points, then $d(x, y) = 0$, so any fixed point is unique. Now, in the Fundamental Inequality, replace x and y by $x_n = f^n(x_0)$ and $x_m = f^m(x_0)$, then

$$\begin{aligned} d(x_n, x_m) &\leq \frac{d(f(x_n), x_n) + d(f(x_m), x_m)}{1 - \alpha} \\ &= \frac{d(f^n(f(x_0)), f^n(x_0)) + d(f^m(f(x_0)), f^m(x_0))}{1 - \alpha} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\alpha^n d(f(x_0), x_0) + \alpha^m d(f(x_0), x_0)}{1 - \alpha} \\
&= \frac{\alpha^n + \alpha^m}{1 - \alpha} d(f(x_0), x_0),
\end{aligned}$$

Since α^k becomes arbitrarily small for k large enough, the sequence $\{x_n\}$ is Cauchy and hence $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f^n(x_0)$ exists. We showed earlier that if such a limit exists, it must converge to a fixed point, say p , which must be unique by our earlier remark. $\square$

If we let $m \rightarrow \infty$ in the last inequality, we get the rate at which $f^n(x_0)$ converges to p :

Corollary 8.5.3 $d(f^n(x_0), p) \leq \frac{\alpha^n}{1 - \alpha} d(f(x_0), x_0).$

Example 8.5.4 Let $f : I \rightarrow I$ where $I \subseteq \mathbb{R}$ is an interval. If $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in I$, where $0 < \alpha < 1$, then f is a contraction mapping, and f has a unique fixed point. The basin of attraction of f is all of I . This is no longer true if $\alpha = 1$ (see the exercises).

The next theorem gives us the notion of *self-similar set*. Here we follow Hutchinson [31] (see also [34]). We use the notion of *compact set*, which we will not define for general metric spaces, but in the case of our main examples, $X = \mathbb{R}^n$ (usually with $n = 2$), a compact set is one that is closed and bounded. Compactness for metric spaces is treated in Chapter 14.

Theorem 8.5.5 (Hutchinson's Theorem) *Let (X, d) be a metric space, and for each $1 \leq i \leq N$, let $f_i : X \rightarrow X$ be a contraction. Then there exists a unique non-empty compact subset K of X that satisfies*

$$K = f_1(K) \cup \cdots \cup f_N(K).$$

K is called the self-similar set with respect to $\{f_1, \dots, f_N\}$.

Before we can prove Theorem 8.5.5 we need some preliminary notions. Define a function on the subsets of X by

$$F(A) = \cup_{i=1}^N f_i(A), \quad \text{for } A \subseteq X,$$

then the idea of the proof is to show the existence of a fixed point of F using the Contraction Mapping Theorem. In order to make this work, we need to turn the set of non-empty subsets of X into a complete metric space. We use the notion of

Hausdorff metric. Set

$$\mathcal{C}(X) = \{A : A \text{ is a non-empty compact subset of } X\},$$

so that $F : \mathcal{C}(X) \rightarrow \mathcal{C}(X)$. We turn this into a metric space $(\mathcal{C}(X), D)$ as follows:

Suppose $A \in \mathcal{C}(X)$ and $\bar{B}_\delta(x) = \{y \in X : d(x, y) \leq \delta\}$ is the closed ball of radius δ centered on x . Then set

$$U_\delta(A) = \cup_{y \in A} \bar{B}_\delta(y) = \{x \in X : d(x, y) \leq \delta \text{ for some } y \in A\}.$$

Think of $U_\delta(A)$ as a closed set that contains A whose boundary lies within δ of A .

Definition 8.5.6 The *Hausdorff metric* D is defined on $\mathcal{C}(X)$ for $A, B \in \mathcal{C}(X)$ by

$$D(A, B) = \inf\{\delta > 0 : A \subseteq U_\delta(B) \text{ and } B \subseteq U_\delta(A)\}.$$

We now show that $D(A, B)$ defines a metric on $\mathcal{C}(X)$, but we omit the proof of completeness:

Theorem 8.5.7 *Let (X, d) be a metric space. Then the Hausdorff metric $D(A, B)$ defines a metric on $\mathcal{C}(X)$. If the metric space (X, d) is complete, then so is the metric space $(\mathcal{C}(X), D)$.*

Proof. From the symmetry of the definition of D we see that $D(A, B) = D(B, A) \geq 0$, and also that $D(A, A) = 0$.

Suppose that $D(A, B) = 0$, then for any $n \in \mathbb{Z}^+$, $A \subseteq U_{1/n}(B)$, so that for any $x \in A$ we can find a sequence $x_n \in B$ with $d(x, x_n) \leq 1/n$. Since B is a closed set (being compact) we have $x \in B$ so that $A \subseteq B$. Similarly $B \subseteq A$, so $A = B$.

To prove the triangle inequality let A, B and C belong to $\mathcal{C}(X)$ and suppose that $D(A, B) < \delta$ and $D(B, C) < \epsilon$, then $U_{\delta+\epsilon}(A) \supseteq C$ and $U_{\delta+\epsilon}(C) \supseteq A$, so that

$$D(A, C) < \delta + \epsilon = D(A, B) + \delta' + D(B, C) + \epsilon',$$

where $\delta' = \delta - D(A, B) > 0$ and $\epsilon' = \epsilon - D(B, C) > 0$ are arbitrary, so that

$$D(A, C) \leq D(A, B) + D(B, C).$$

□

We now show that a contraction on the metric space (X, d) gives rise to a contraction on $(\mathcal{C}(X), D)$.

Proposition 8.5.8 *Let $f : X \rightarrow X$ be a contraction mapping on (X, d) with*

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y), \quad (0 < \alpha < 1), \quad \text{for all } x, y \in X.$$

Then

$$D(f(A), f(B)) \leq \alpha \cdot D(A, B) \quad \text{for all } A, B \in \mathcal{C}(X).$$

Proof. If $\delta > D(A, B)$ then $U_\delta(A) \supseteq B$, so if $x \in B$ there exists $y \in A$ with $d(x, y) \leq \delta$. Since f is a contraction, this implies $d(f(x), f(y)) \leq \alpha d(x, y) \leq \alpha\delta$. Consequently

$$U_{\alpha\delta}(f(A)) \supseteq f(B).$$

In a similar way $U_{\alpha\delta}(f(B)) \supseteq f(A)$, so that $D(f(A), f(B)) \leq \alpha\delta$ and the result follows. $\square$

Proof of Theorem 8.5.5 We first show that if $A_1, A_2, B_1, B_2 \in \mathcal{C}(X)$, then

$$D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}.$$

To see this, suppose that $\delta > \max\{D(A_1, B_1), D(A_2, B_2)\}$, then

$$\delta > D(A_1, B_1) \geq \inf\{\delta' : U_{\delta'}(A_1) \supseteq B_1\},$$

so in particular

$$U_\delta(A_1) \supseteq B_1 \quad \text{and} \quad U_\delta(A_2) \supseteq B_2.$$

This clearly implies $U_\delta(A_1 \cup A_2) \supseteq B_1 \cup B_2$. A similar argument implies that $U_\delta(B_1 \cup B_2) \supseteq A_1 \cup A_2$, so that $\delta > D(A_1 \cup A_2, B_1 \cup B_2)$ and it follows that $D(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{D(A_1, B_1), D(A_2, B_2)\}$.

Now using this result repeatedly we have

$$D(F(A), F(B)) = D(\cup_{j=1}^N f_j(A), \cup_{j=1}^N f_j(B)) \leq \max_{1 \leq j \leq N} D(f_j(A), f_j(B)).$$

By Proposition 8.5.5, $D(f_i(A), f_i(B)) \leq \alpha_i D(A, B)$, where α_i are the contraction constants for the maps f_i . If $\alpha = \max_{1 \leq i \leq N} \alpha_i$, then

$$D(F(A), F(B)) \leq \alpha D(A, B),$$

so that F is a contraction on $(\mathcal{C}(X), D)$, a complete metric space. The result follows by the contraction mapping theorem. $\square$

Examples 8.5.9 Take $X = \mathbb{R}$ and denote by C the Cantor middle-thirds set. If we define $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$, by

$$f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x}{3} + \frac{2}{3},$$

then clearly these are contractions on $\mathbb{R}$, and then we define F on $\mathcal{C}(\mathbb{R})$ by $F(A) = f_1(A) \cup f_2(A)$ for all compact subsets $A \subset \mathbb{R}$. According to Hutchinson's Theorem, if J is any compact subset of $\mathbb{R}$, $\lim_{n \rightarrow \infty} F^n(J) = K$ is a unique fixed point of F in $\mathcal{C}(\mathbb{R})$. It is easy to check that C is a fixed point of F , so we must have $K = C$, the Cantor set.

2. Set $X = \mathbb{R}^2$ and let (a_j, b_j) , $j = 1, 2, 3$ be the vertices of an equilateral triangle. Define contractions $f_j : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+a_1}{2} \\ \frac{y+b_1}{2} \end{pmatrix}, \quad f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+a_2}{2} \\ \frac{y+b_2}{2} \end{pmatrix}, \quad f_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+a_3}{2} \\ \frac{y+b_3}{2} \end{pmatrix}.$$

Then we shall see that $F = f_1 \cup f_2 \cup f_3$ gives rise to the Sierpinski triangle as its unique fixed point.

Exercises 8.5

1. We have used the Intermediate Value Theorem throughout these notes in the following form as one of our main tools: *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. If $f(a) < 0$ and $f(b) > 0$ then there exists $c \in (a, b)$ with $f(c) = 0$.* Prove the Intermediate Value Theorem using the following steps (our proof depends on the Monotone Convergence Theorem for real sequences - see the Appendix):

(a) Use the Bisection Method to obtain a sequence of nested intervals $[a_n, b_n]$ of length $(b-a)2^{-n}$ where $f(a_n) < 0$ and $f(b_n) \geq 0$. (I.e., subdivide $[a, b]$ and set $a_1 = (a+b)/2$ if $f(a+b)/2 < 0$, otherwise set it equal to b_1 , and continue in this way.)

(b) Show that $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ exist and are equal.

(c) Use the continuity of f to conclude that

$$f \left(\lim_{n \rightarrow \infty} a_n \right) \leq 0 \quad \text{and} \quad f \left(\lim_{n \rightarrow \infty} b_n \right) \geq 0.$$

(d) Conclude that $c = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ satisfies $f(c) = 0$.

2. Let $f : X \rightarrow X$ be a contraction mapping on a metric space (X, d) . Prove that f is continuous.

3. Let $f : I \rightarrow I$ where I is a non-empty subinterval of $\mathbb{R}$. If f has the property

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x, y \in I.$$

Note that $f(x)$ is necessarily continuous. Show that:

- (a) f has at most one fixed point.
- (b) The function $f(x) = x + 1/x$ has this property on $[1, \infty)$, but has no fixed points.
- (c) The function $g(x) = (x + \sin(x))/2$ has this property on $\mathbb{R}$. Does g have a fixed point?
- (d) Show that if f has this property on $I = [a, b]$, ($a < b$), then $f(x)$ has a unique fixed point. (Hint: Consider $\inf\{|f(x) - x| : x \in [a, b]\}$).

4. Let $f(x) = x^2 - a$ with $1 < a < 3$, and let N_f be the corresponding Newton function. Show that N_f satisfies the hypothesis of the Contraction Mapping Theorem on $(1, \infty)$. What is the fixed point?

5. Brouwer's Fixed Point Theorem says that if we have a continuous map $f : D \rightarrow D$, where D is a closed ball in $\mathbb{R}^n$, then f has a fixed point in D (e.g. $D = [0, 1] \subseteq \mathbb{R}$, or $D = \{(x, y); x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$). Give an example of a continuous onto map $f : (0, 1) \rightarrow (0, 1)$ which does not have a fixed point.