

Q1

$$x' = y$$

$$y' = x^3 - x$$

We equate the right hand side to zero

$$y = 0 \quad \text{and} \quad x^3 - x = 0$$

$$\text{Since } x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$$

$\therefore$ critical points are :

$$(0, 0), (1, 0) \text{ and } (-1, 0)$$

The corresponding Jacobian matrix is

$$A_1 = g'(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Since $\det A_2 < 0$, critical points $(1, 0)$ and $(-1, 0)$ are both saddle points. The eigen values of matrix A_1 are $\pm i$, and ~~according~~ the critical point at $(0, 0)$ remains in doubt

Q2

$$x' = x - y - x^2 + xy \Rightarrow f(x, y)$$

$$y' = -x^2 - y \Rightarrow g(x, y)$$

Critical points are given by

$$x' = 0 \Rightarrow x - y - x^2 + xy = 0 \dots (i)$$

$$y' = 0 \Rightarrow -x^2 - y = 0 \dots (ii)$$

substituting (ii) in (i) we get

$$a) \quad x^2 + x^2 - x^2 + x(-x^2) = 0$$

$$x - x^3 = 0 \Rightarrow x(x+1)(x-1) = 0$$

The critical points are $(0, 0)$, $(1, 0)$, $(-1, 0)$

$$\text{Jacobian matrix } J = \begin{pmatrix} 1 - 2x - y & x - 1 \\ -2x & -1 \end{pmatrix}$$

$\therefore$ The linear system near the critical point $A(0, 0)$ is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = J|_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

linear system near $(-1, -1)$ is

$$b) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = J|_{(-1,-1)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

linear system near $(1, -1)$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = J|_{(1,-1)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

(c)

The eigen values are 1 and -1

The critical point $(0, 0)$ is a saddle point

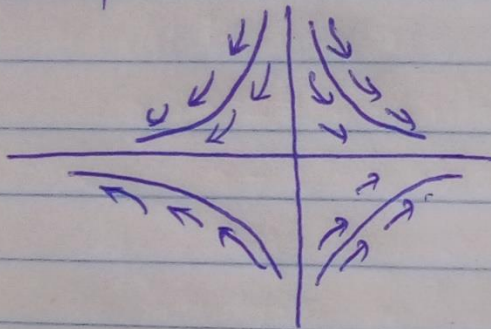
$$J(-1, -1) = \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix}, \quad \lambda = \frac{1 \pm \sqrt{17}i}{2}$$

The critical point $(-1, -1)$ is unstable point spiral

$$J(1, -1) = \begin{pmatrix} -2 & 0 \\ -2 & -1 \end{pmatrix}$$

The eigen values are $-2, -1$, node is unstable

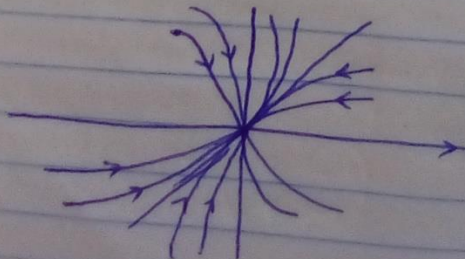
d) Phase portrait near $(0, 0)$



Phase portrait near $(-1, -1)$



Phase portrait near $(1, -1)$



Q3

$$\begin{aligned}x' &= -ky + x(1-x^2-y^2) \\y' &= kx + y(1-x^2-y^2)\end{aligned}$$

soln

$$f(x,y) = x' = -ky + x - x^3 - xy^2$$

$$g(x,y) = y' = kx + y - x^2y - y^3$$

$$\frac{\partial f}{\partial x} = 1 - 3x^2 - y^2, \quad \frac{\partial f}{\partial y} = -k - 2xy$$

$$\frac{\partial g}{\partial x} = k - 2xy, \quad \frac{\partial g}{\partial y} = 1 - x^2$$

at $(0,0)$

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}$$

the characteristic function

$$\therefore (1-x)(1-x) + k^2 = 0$$

$$\Rightarrow 1 - 2x + x^2 + k^2 = 0$$

$$x^2 - 2x + k^2 + 1 = 0$$

$$\text{solving } x = \frac{2 \pm \sqrt{4 - 4(k^2 + 1)}}{2}$$

$$= 1 \pm ik$$

When $k > 0$ or $k < 0$, eigenvalues are complex with positive parts. So $(0,0)$ is unstable

When $k=0$, we have two positive eigen values hence here $(0,0)$ is unstable node