P26-P36

Boolean Algebra

Special Algebra where the $\overset{x, y, z}{\text{variables}}$ can be $\overset{\text{False}}{0}$ or $\overset{\text{True}}{1}$

operators :

canonical operators

AND $x \cdot y$ $x * y$ xy → read as x and y

OR $x + y$ → reads as x or y

NOT $\bar{x}$ x' x^c $\sim x$ $\neg x$ → read as not x

2^n rows n= number of vars

Truth Table

AND

	x	y	$x \cdot y$
0	0	0	0
1	0	1	0
2	1	0	0
3	1	1	1

OR

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

Not

x	$\bar{x}$	$\bar{\bar{x}}$
0	1	0
1	0	1

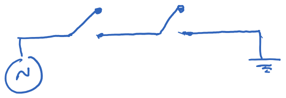
=

$x = \bar{\bar{x}}$

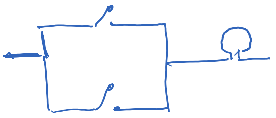
Double negation

index	x	y	z
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

and



OR



Not looked at

Short-circuit if ($\underset{A}{\overset{\text{True}}{\boxed{}}}$ $\overset{\text{or}}{||}$ $\underset{B}{\boxed{}}$ || $\underset{C}{\boxed{}}$ || $\underset{D}{\boxed{}}$)

Not looked at

AND : if ($\overset{\text{False}}{\boxed{}}$ && $\boxed{}$ && ..)

A and B and C

Construct Truth Table for $F = \underbrace{\overline{B} + C}\boxed{+}\underbrace{ABC}$ 3var → $2^3 = 8$ rows

A	B	C	$\overline{B}$	$\overline{B}+C$	ABC	F
0	0	0	1	1	0	1
0	0	1	1	1	0	1
0	1	0	0	0	0	0
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	0	1	1	1	0	1
1	1	0	0	0	0	0
1	1	1	0	1	1	1

Construct Truth Table for $Z = \overline{A}\left(A+B\right)+\left(A+B\right)\left(A+\overline{B}\right)+C$

A	B	C	$\bar{A}$	$\bar{B}$	$A+B$	$\bar{A}(A+B)$	$(A+\bar{B})$	$(A+B)(A+\overline{B)}$	Z
0	0	0							
0	0	1							
0	1	0							
0	1	1							
1	0	0							
1	0	1							
1	1	0							
1	1	1							

Prove using truth table $\overbrace{xy+\bar{x}z+yz}^{LS}=\overbrace{xy+\bar{x}z}^{RS}$ 3 var $\rightarrow$ 8 rows

x	y	z	$\bar{x}$	xy	$\bar{x}z$	yz	LS	RS	LS $\overset{?}{=}$ RS
0	0	0	1	0	0	0	0	0	Yes
0	0	1	1	0	1	0	1	1	Yes
0	1	0	1	0	0	0	0	0	Yes
0	1	1	1	0	1	1	1	1	Yes
1	0	0	0	0	0	0	0	0	Yes
1	0	1	0	0	0	0	0	0	Yes
1	1	0	0	1	0	0	1	1	Yes
1	1	1	0	1	0	1	1	1	Yes

Examine the validity of $\overset{A}{\underbrace{(\bar{x}+y+z)}}\overset{B}{\underbrace{(x+\bar{y}\,\bar{z})}}=\underbrace{x\,y+\bar{y}\,z}_{Rs}$

index	x	y	z	$\bar{x}$	$\bar{y}$	$\bar{z}$	A	$\bar{y}\bar{z}$	B	LS	xy	$\bar{y}z$	Rs
0	0	0	0	1	1	1	1	1	1	$\boxed{1}$	0	0	$\boxed{0}$
1	0	0	1	1	1	0	1	0	0	$\boxed{0}$	0	1	$\boxed{1}$
2	0	1	0	1	0	1	1	0	0	0	0	0	0
3	0	1	1	1	0	0	1	0	0	0	0	0	0
4	1	0	0	0	1	1	0	1	1	0	0	0	0
5	1	0	1	0	1	0	1	0	1	1	0	1	1
6	1	1	0	0	0	1	1	0	1	1	1	0	1
7	1	1	1	0	0	0	1	0	1	1	1	0	1

$\neq$

LS $\neq$ RS

B
~~E~~
~~D~~
M
A
~~S~~

Order of operations

1) Brackets
2) Negation
3) AND
4) OR

Example: Is $a + a'b$ equal to $1.b$? Why? Note $a + a' = 1$

$a + a'b$ No, AND is prioritized over OR

$a + a'b$ = 14d = $a + b$

Simplifying using theorems

step statement Reason

#

Simplify $Z = \bar{A}(A+B)+(A+B)(A+\bar{B})+C$

STEPS	statement	POSTULATE OR THEOREM
1)	$\underbrace{\bar{A}(A+B)}+(A+B)(A+\bar{B})+C$	Given
2)	$\bar{A}B + \underbrace{(A+B)(A+\bar{B})}+C$	[14c]
3)	$\bar{A}B + A + \underbrace{(B \cdot \bar{B})} + C$	[8b]
4)	$\bar{A}B + \underbrace{A + 0} + C$	[12a]
5)	$\underbrace{\bar{A}B + A} + C$	[9b]
optional → 6)	$\underbrace{A + \bar{A}B} + C$	[6 b]
7)	$A+B + C$	[14d]

The optimal (simplest form) $Z = A+B+C$

Prove the following:

LS $\underbrace{A\left(A+\overline{B}\right)+\overline{B}\left(B+C\right)}_{}=\underbrace{A+\overline{B}C}_{}$ RS

STEPS	POSTULATE OR THEOREM
1) $LS = \underbrace{A(A+\overline{B})}_{\downarrow} + \overline{B}(B+C)$	Given
2) $A + \underbrace{\overline{B}(B+C)}_{\downarrow}$	[14b]
3) $A + \overline{B}\cdot C = RS$	[14c]

Hence proved ☺

$LS = AB\overline{C}+A\overline{B}\,\overline{C}+\overline{A}B\overline{C}=\overline{C}\left(B+A\right) = Rs$

STEPS	POSTULATE OR THEOREM
1) $LS = AB\overline{C} + A\overline{B}\overline{C} + \overline{A}B\overline{C}$	Given
2) $= (\underbrace{AB + A\overline{B}}_{} + \overline{A}B)\cdot\overline{C}$	[8a]
3) $(A\cdot\underbrace{(B+\overline{B})}_{\downarrow} + \overline{A}B)\cdot\overline{C}$	[8a]
4) $(\underbrace{A\cdot 1}_{\downarrow} + \overline{A}B)\cdot\overline{C}$	[12b]
5) $(\underbrace{A + \overline{A}B}_{\downarrow})\cdot\overline{C}$	[9a]
6) $(A+B)\cdot\overline{C}$	[14d]
7) $\overline{C}\cdot(A+B) = RS$	[6a]

☺

$(A+\overline{B})(A+B)=A$

STEPS	POSTULATE OR THEOREM
1) $LS=(A+\overline{B})(A+B)$	Given
2) $A+(\overline{B}\cdot B)$	[8b]
3) $A+0$	[12a]
4) $A=RS$	[9b]

$A\overline{B}(A\overline{B}+B\overline{C})=A\overline{B}$

STEPS	POSTULATE OR THEOREM
1) LS $\underbrace{A\overline{B}}_{x}(\underbrace{A\overline{B}}_{x}+\underbrace{B\overline{C}}_{y})$	Given
2) $A\overline{B}=RS$	[14b] → $x(x+y)=x$ $\square(\square+\triangle)$

$$(xy + ABC)(xy + \bar{A} + \bar{B} + \bar{C}) = xy$$

STEPS	POSTULATE OR THEOREM
1) $LS = (xy + ABC)(xy + \bar{A} + \bar{B} + \bar{C})$	Given
2) $= xy + \left((ABC) \cdot (\underbrace{\bar{A} + \bar{B} + \bar{C}})\right)$	[8b] // factored $xy+$ out
3) $= xy + \left((ABC) \cdot (\overline{ABC})\right)$	[DM]
4) $= xy + 0$	[12a]
5) $= xy = RS$	[9b]

$$\overbrace{A(\bar{A} + B) + B(B + C) + B}^{LS} = \underbrace{B}_{RS}$$

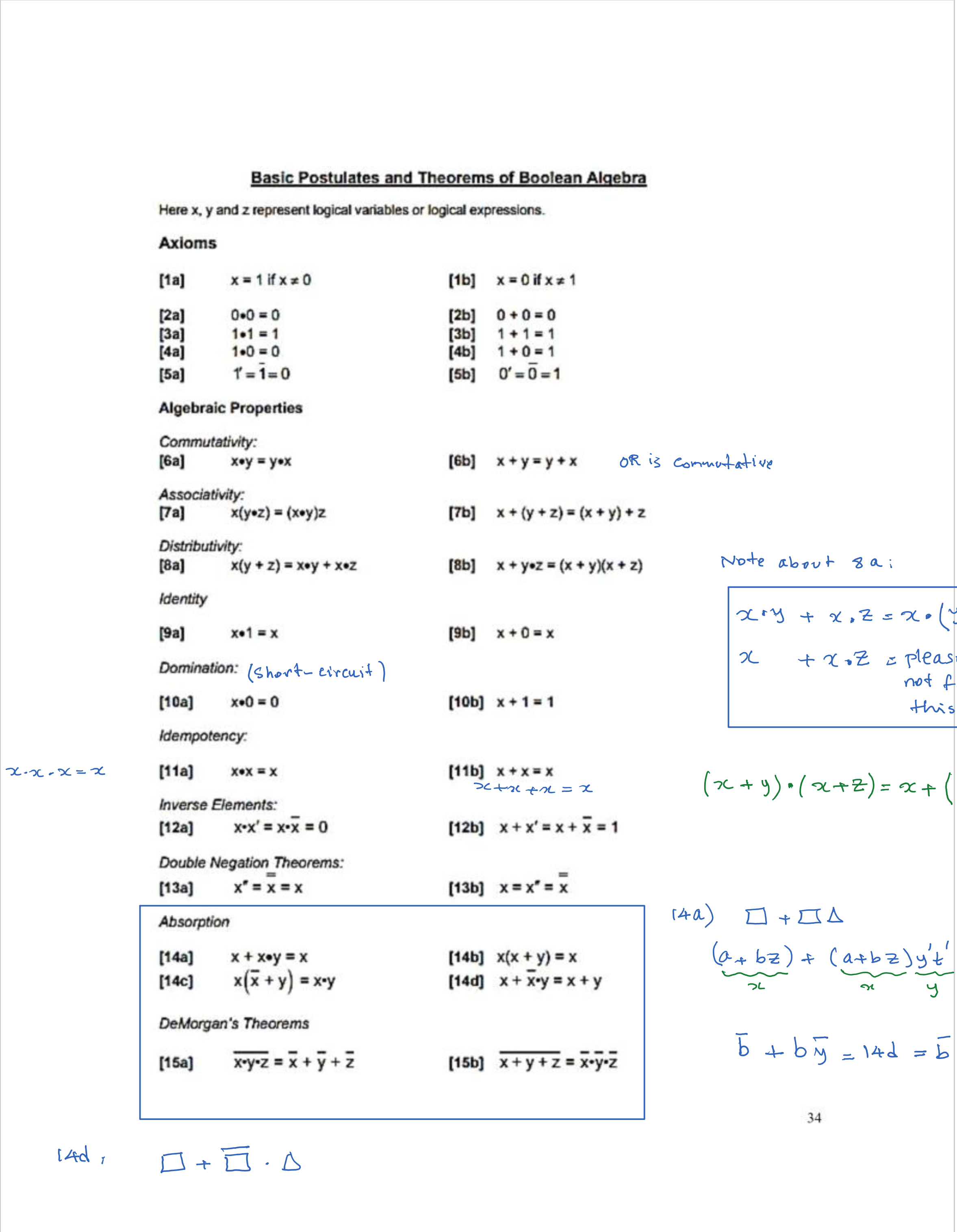
STEPS	POSTULATE OR THEOREM
1) $LS = \underbrace{A(\bar{A} + B)} + B(B + C) + B$	Given
2) $= AB + \underbrace{B(B+C)} \underbrace{+B}$	[14c]
3) $= AB + B$	[14a]
4) $= B$	[14a]

$AB + B$

$\neq B(A+1)$ please don't do this

Simplify $AB\left(\bar{A}CD + AE + EG\right)$

STEPS		POSTULATE OR THEOREM	
1)	$AB(\bar{A}CD + AE + EG)$	Given	
2)	$AB\bar{A}CD + ABAE + ABEG$	[8a]	
2.5)	$\underbrace{A\bar{A}}_{\downarrow}BCD + AABE + ABEG$	[6a]	// Optional step
3)	$\underbrace{0BCD}_{\downarrow} + AABE + ABEG$	[12a]	
4)	$0 + AABE + ABEG$	[10a]	
5)	$\underbrace{AA}_{\downarrow}BE + ABEG$	[9b]	
6)	$ABE + ABEG$	[11a]	
7)	ABE	[14a]	



Basic Postulates and Theorems of Boolean Algebra

Here x, y and z represent logical variables or logical expressions.

Axioms

[1a] $x = 1$ if $x \neq 0$

[1b] $x = 0$ if $x \neq 1$

[2a] $0 \bullet 0 = 0$

[2b] $0 + 0 = 0$

[3a] $1 \bullet 1 = 1$

[3b] $1 + 1 = 1$

[4a] $1 \bullet 0 = 0$

[4b] $1 + 0 = 1$

[5a] $1' = \overline{1} = 0$

[5b] $0' = \overline{0} = 1$

Algebraic Properties

Commutativity:

[6a] $x \bullet y = y \bullet x$

[6b] $x + y = y + x$ OR is Commutative

Associativity:

[7a] $x(y \bullet z) = (x \bullet y)z$

[7b] $x + (y + z) = (x + y) + z$

Distributivity:

[8a] $x(y + z) = x \bullet y + x \bullet z$

[8b] $x + y \bullet z = (x + y)(x + z)$

Note about 8a:

$$\boxed{\begin{array}{l} x \cdot y + x \cdot z = x \cdot (y + z) \\ x + x \cdot z = \text{please do not factor this.} \end{array}}$$

Identity

[9a] $x \bullet 1 = x$

[9b] $x + 0 = x$

Domination: (Short-circuit)

[10a] $x \bullet 0 = 0$

[10b] $x + 1 = 1$

Idempotency:

$x \cdot x \cdot x = x$

[11a] $x \bullet x = x$

[11b] $x + x = x$

$x + x + x = x$

$(x + y) \cdot (x + z) = x + (y \cdot z)$

Inverse Elements:

[12a] $x \cdot x' = x \cdot \overline{x} = 0$

[12b] $x + x' = x + \overline{x} = 1$

Double Negation Theorems:

[13a] $x'' = \overline{\overline{x}} = x$

[13b] $x = x'' = \overline{\overline{x}}$

Absorption

[14a] $x + x \bullet y = x$

[14b] $x(x + y) = x$

[14c] $x(\overline{x} + y) = x \cdot y$

[14d] $x + \overline{x} \cdot y = x + y$

DeMorgan's Theorems

[15a] $\overline{x \cdot y \cdot z} = \overline{x} + \overline{y} + \overline{z}$

[15b] $\overline{x + y + z} = \overline{x} \cdot \overline{y} \cdot \overline{z}$

14a) $\square + \square \triangle$

$\underbrace{(a + bz)}_{x} + \underbrace{(a + bz)}_{x}\underbrace{y't'}_{y}$

$\overline{b} + b\overline{y} = 14d = \overline{b} + \overline{y}$

34

14d, $\square + \overline{\square} \cdot \triangle$

$$\overline{x+y} = \bar{x} \cdot \bar{y}$$

$$\overline{x \cdot y} = \bar{x} + \bar{y}$$

De Morgan's Theorem

$2^2 = 4$

$\overline{x+y+z} = \bar{x} \cdot \bar{y} \cdot \bar{z}$

=

x	y	$\bar{x}$	$\bar{y}$	xy	$\overline{xy}$	$x+y$	$\overline{x+y}$	$\bar{x}+\bar{y}$	$\bar{x}\bar{y}$
0	0	1	1	0	1	0	1	1	1
0	1	1	0	0	1	1	0	1	0
1	0	0	1	0	1	1	0	1	0
1	1	0	0	1	0	1	0	0	0

=

Simplify $\overline{\bar{B}+\bar{C}+BC}$

STEPS	POSTULATE OR THEOREM
1) $\overline{\underbrace{\bar{B}+\bar{C}} + BC}$	Given
2) $\overline{\underbrace{\overline{BC} + BC}}$	[DM]
3) $\bar{1}$	[12b]
4) 0	[5a]

$\bar{B} + \bar{C} = \overline{B \cdot C}$

Simplify $\overline{AB\overline{C}+\overline{A}BC}$

STEPS		POSTULATE OR THEOREM
1)	$\overline{AB\overline{C}+\overline{A}BC}$	Given
2)	$\overline{AB\overline{C}}\bullet\overline{\overline{A}BC}$	DM
3)	$(\overline{A}+\overline{B}+\overline{\overline{C}})\cdot(\overline{\overline{A}}+\overline{B}+\overline{C})$	DM
4)	$(\overline{A}+\overline{B}+C)\cdot(A+\overline{B}+\overline{C})$	[13a]
5)	$\overline{B}+(\overline{A}+C)\cdot(A+\overline{C})$	[8b]

Identify Inverse elements

$x+\overline{x}=1$ $x\cdot\overline{x}=0$

x	$\overline{x}$	Yes
ab	$\overline{ab}$	Yes
ab	$\overline{a}\overline{b}$	No
ab	$\overline{a}+\overline{b}$	Yes
$a+b$	$\overline{\overline{a}\overline{b}}$	No
$\overline{a}+\overline{b}$	$\overline{a+b}$	no
$a\overline{b}$	$\overline{a}b$	no
$a\overline{b}$	$\overline{a\overline{b}}$	Yes
$a\overline{b}$	$\overline{a}+b$	Yes

$\overline{\overline{a}\cdot\overline{b}}=\overline{\overline{a}}+\overline{\overline{b}}$

$=a+b$

Simplify $\overline{\overline{\overline{\overline{a}+b}\cdot\overline{\overline{a}+\overline{b}+c}}\cdot\overline{\overline{b}+\overline{\overline{b}+c}}}$ $\overline{\Box\cdot\Box}$

STEPS		POSTULATE OR THEOREM
1)	Repeat	Given
2)	$\overline{\overline{\overline{\overline{a}+b}\cdot\overline{\overline{a}+\overline{b}+c}}}+\overline{\overline{\overline{b}+\overline{\overline{b}+c}}}$	DM
3)	$\left(\overline{\overline{a}+b}\bullet\overline{\overline{a}+\overline{\overline{b}+c}}\right)+\left(\overline{b}+\overline{\overline{b}+c}\right)$	[13a]
4)	$\left(\overline{\overline{a}}\cdot\overline{b}\bullet\overline{\overline{a}}\cdot\overline{\overline{\overline{b}+c}}\right)+\left(\overline{b}+\overline{\overline{b}}\cdot\overline{c}\right)$	DM
5)	$\left(a\cdot\underbrace{\overline{b}\bullet a}\cdot(\overline{b}+c)\right)+\left(\overline{b}+b\overline{c}\right)$	[13a]
6)	$\left(\underbrace{a\cdot a}\cdot\overline{b}\cdot(\overline{b}+c)\right)+\left(\overline{b}+b\overline{c}\right)$	[6a]
7)	$\left(a\cdot\underbrace{\overline{b}\cdot(\overline{b}+c)}\right)+\left(\overline{b}+b\overline{c}\right)$	[11a]
8)	$(a\cdot\overline{b})+(\overline{b}+\underbrace{b\overline{c}})$	[14b]
9)	$\underbrace{a\overline{b}+\overline{b}}+\overline{c}$	[14d]
10)	$\overline{b}+\overline{c}$	[14a]

$\overline{b}+\overline{c}=\overline{bc}$

SOP and POS forms

POS: product of sum

SOP: Sum of product

$$F=\underbrace{x'yz}_{}+\underbrace{x'y}_{}+\underbrace{x'y'z'}_{}+\underbrace{xyz}_{}=\text{SOP form}$$

$x'yz$: 0 1 1

$x'y$: 0 1 1, 0 1 0

$x'y'z'$: 0 0 0

xyz: 1 1 1

3 , 2 , 3 , 0 , 7

= Minterm

$= \sum m\,(0, 2, 3, 7)$

= locations with "1" outputs

index	x	y	z	
	0	1	1	$\boxed{}$ ←

$$S = \underbrace{x\,y\,z'}_{\substack{\downarrow\ \downarrow\ \downarrow\\ 1\ \ 1\ \ 0}} + \underbrace{x\,z}_{\substack{\downarrow\\ 1\ 0\ 1\\ 1\ 1\ 1}} + \underbrace{x'\,y'}_{\substack{\downarrow\\ 0\ 0\ 0 \leftarrow 0\\ 0\ 0\ 1 \leftarrow 1}}$$

$$\text{SOP} = \sum m\,(\underbrace{0, 1, 5, 6, 7}_{\uparrow})$$

where we have 1's in the output

row	x	y	z	F
0				1
1				0
2				1
3				1
4				0
5				0
6				0
7				1

$\text{Maxterm} = \prod M\,(2, 3, 4)$

= where the 0's are

$$G = \overset{0\ 1\ 0}{(x+y'+z)} \cdot \overset{\substack{0\ 0\ 0\\ 0\ 0\ 1}}{(x+y)} \cdot \overset{0\ 1\ 1}{(x+y'+z')} \cdot \overset{1\ 1\ 0}{(x'+y'+z)}$$ = POS form

Maxterm $= \prod M(\underbrace{0, 1, 2, 3, 6}_{\text{where the zeros are}})$

output

x	y	z	G	
			0	
			0	
			0	
			0	
1	0	0	$\boxed{1}$	← $xy'z'$
1	0	1	$\boxed{1}$	← $xy'z$
			0	
1	1	1	$\boxed{1}$	← xyz

Minterm $= \sum m(4, 5, 7)$

$G =$ SOP $= xy'z' + xy'z + xyz$