

1 Introduction to Portfolio Theory

1.1 Introduction

Portfolio theory considers how best to divide resources over a given collection of assets.

A number of different asset characteristics can be considered, in particular in relation to the level of return and risk.

The best division of resources, the portfolio, depends upon the objectives of the investor.

This could be to maximize return for a given level of risk, or to minimise risk for a given level of return, for instance.

The riskiness of a given portfolio is measured in terms of the variability of the possible return.

To minimise risk, we often wish to diversify our portfolio, so that it contains a range of assets of different types.

We shall use optimization methods to find the answer to these problems.

1.2 Rates of return

In our portfolios we shall consider two properties of an asset, risk and rate of return.

For an asset with value $S(0)$ at time 0 and $S(T)$ at time T , the rate of return ρ is given by

$$S(T) = (1 + \rho)S(0).$$

Equivalently

$$\rho = \frac{S(T) - S(0)}{S(0)}.$$

The rate or return can be thought of as equivalent to the interest rate on a bank account necessary for an investment of $S(0)$ to grow to $S(T)$ over a time period of length T .

For a bank account ρ and $S(T)$ are effectively fixed, so that they are known at time 0, but for most investments there is some degree of uncertainty. Thus $S(T)$ is unknown, and follows some distribution i.e. it is a random variable. Thus ρ is also a random variable.

A random variable (usually) has an expectation or mean, and we define

$$\mu = E[\rho]$$

as the *expected rate of return*. Similarly a random variable (usually) has a variance, and we define

$$\sigma^2 = Var(\rho) = E[(\rho - \mu)^2].$$

The larger σ^2 , the more “risky” the asset is.

Suppose that we have a collection of n assets $\{S_1, S_2, \dots, S_n\}$. Here S_i is the label for the asset, but we will also use $S_i(t)$ as the value of S_i at time t .

We denote the rate of return and variance of asset S_i by μ_i and σ_i^2 respectively.

It is important to consider the relationship between different assets. In particular how returns on any pair of assets influence each other.

To do this we consider the covariance of two assets S_i and S_j ,

$$\sigma_{ij} = E[(\rho_i - \mu_i)(\rho_j - \mu_j)].$$

Note that $\sigma_{ij} = \sigma_{ji}$, and $\sigma_{ii} = \sigma^2$.

The relationships between n assets are summarised in the variance-covariance matrix

$$V = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{pmatrix}.$$

A portfolio is an investment of value W divided over the n assets. Thus we allocate value W_i to S_i where

$$\sum_{i=1}^n W_i = W.$$

It is convenient to work in terms of the proportions of a portfolio rather than absolute values, so we consider the proportion invested in S_i , $w_i = W_i/W$, where

$$\sum_{i=1}^n w_i = 1.$$

If there is no reallocation of assets, a portfolio P is chosen by the set of initial investments $S_i(0)$.

The value of the portfolio at time t , $Q_P(t)$, is

$$Q_P(t) = \sum_{i=1}^n \frac{W_i}{S_i(0)} S_i(t).$$

(note that of course $Q_P(0) = W$).

The rate of return ρ_P of our portfolio P can be calculated in terms of the rates of return and weights of the component assets as follows.

$$\begin{aligned}
 \rho_P(t) &= \frac{Q_P(t) - Q_P(0)}{Q_P(0)} = \\
 &= \frac{\sum_{i=1}^n \frac{W_i}{S_i(0)} S_i(t) - W}{W} = \\
 &= \sum_{i=1}^n \frac{W_i}{W} \frac{S_i(t)}{S_i(0)} - \sum_{i=1}^n \frac{W_i}{W} = \\
 &= \sum_{i=1}^n w_i \frac{S_i(t) - S_i(0)}{S_i(0)} = \\
 &= \sum_{i=1}^n w_i \rho_i.
 \end{aligned}$$

Thus the rate of return for a portfolio is the weighted average of the rates of returns for the individual investments, where the weights are the proportions invested in each asset.

The expected rate of return is given by

$$\mu_P = E \left[\sum_{i=1}^n w_i \rho_i \right] = \sum_{i=1}^n w_i \mu_i.$$

The variance of the rate of return of the portfolio can be calculated in a similar way, as shown below.

$$\begin{aligned}
\sigma_P^2 &= E[(\rho_P - \mu_P)^2] = \\
&E \left[\left(\sum_{i=1}^n w_i \rho_i - \sum_{i=1}^n w_i \mu_i \right)^2 \right] = \\
&E \left[\sum_{i=1}^n (w_i (\rho_i - \mu_i))^2 \right] = \\
&E \left[\left(\sum_{i=1}^n w_i (\rho_i - \mu_i) \right) \left(\sum_{i=1}^n w_i (\rho_i - \mu_i) \right) \right] = \\
&\sum_{i,j=1}^n w_i w_j E[(\rho_i - \mu_i)(\rho_j - \mu_j)] = \\
&\sum_{i,j=1}^n w_i w_j \sigma_{ij} = \\
&\mathbf{w}^T V \mathbf{w},
\end{aligned}$$

where $\mathbf{w}^T = (w_1, w_2, \dots, w_n)$.

Example 1

Consider a portfolio with three assets S_1, S_2 and S_3 which have expected rates of return 0.1, 0.15 and 0.2 respectively, and variance-covariance matrix

$$V = \begin{pmatrix} 0.1 & 0.1 & -0.1 \\ 0.1 & 0.2 & 0.1 \\ -0.1 & 0.1 & 0.6 \end{pmatrix}.$$

Find the expected rates of return and the variances of the following portfolios, where the vector entries are the weights of the three assets in order.

- (i) (0, 1, 0)
- (ii) (0, 0, 1)
- (iii) (0.5, 0, 0.5)
- (iv) (0.3, 0.4, 0.3)

comment on your solutions.

SOLUTION

2 Markowitz theory

2.1 Introduction

There are a number of factors that may influence what makes a good portfolio.

We will concentrate only on the desired level of return and the tolerance of risk of our investor.

Generally the higher the potential risk, the greater the return needed to compensate for it.

Investors in small companies that might go bankrupt would require a higher expected return than investors in a large stable company.

Thus when two investments are available if they have the same rate of return, the one with the smaller variance is preferable.

If they have the same variance, the one with the higher rate of return is preferable.

If one has a higher rate of return, and the other the lower variance, which is better depends upon the tolerance of risk of the investor.

We shall try to find a portfolio that for a given rate of return μ_P , has the smallest variance σ_P^2 .

This is the *Markowitz problem*.

We state this mathematically as follows:

We wish to minimise

$$f(w_1, \dots, w_n) = \frac{1}{2} \sum_{i,j=1}^n w_i w_j \sigma_{ij},$$

(the 1/2 just makes subsequent expressions tidier) subject to

$$g_1(w_1, \dots, w_n) = \sum_{i=1}^n w_i \mu_i - \mu_P = 0,$$

and

$$g_2(w_1, \dots, w_n) = \sum_{i=1}^n w_i - 1 = 0.$$

We thus minimise an *objective function* subject to two constraints. The first constraint says that the expected rate of return is set to μ_P . The second constraint simply says that the sum of the investment weights adds to 1.

To solve this, we use the Lagrange Multiplier method.

2.2 Lagrange multipliers

The method of Lagrange multipliers for our constrained problem involves solving a related unconstrained problem.

We consider the following function involving our three terms f, g_1 and g_2 , called the Lagrangian:

$$L(w_1, \dots, w_n | \lambda_1, \lambda_2) = f(w_1, \dots, w_n) - \lambda_1 g_1(w_1, \dots, w_n) - \lambda_2 g_2(w_1, \dots, w_n).$$

A critical point (alternatively called an equilibrium point) of the Lagrangian is where all of its derivatives (with respect to $w_1, w_2, \dots, w_n, \lambda_1$ and λ_2) are zero.

This occurs precisely at a critical point of f (where all of its derivatives, with respect to $w_1, w_2, \dots, w_n$, are zero) which satisfies the two constraints $g_1 = 0, g_2 = 0$.

To find our optimal portfolio, we thus need to take all of the derivatives of the Lagrangian and set each of them to be equal to 0.

In particular we note that, as mentioned above, we must have

$$\begin{aligned}\frac{\partial L}{\partial \lambda_1} &= -g_1(w_1, \dots, w_n) = 0, \\ \frac{\partial L}{\partial \lambda_2} &= -g_2(w_1, \dots, w_n) = 0.\end{aligned}$$

2.3 The case with two assets

For two assets we have the following simultaneous equations:

$$\begin{aligned}\frac{\partial L}{\partial w_1} &= w_1\sigma_1^2 + w_2\sigma_{12} - \lambda_1\mu_1 - \lambda_2 = 0, \\ \frac{\partial L}{\partial w_2} &= w_2\sigma_2^2 + w_1\sigma_{12} - \lambda_1\mu_2 - \lambda_2 = 0, \\ \frac{\partial L}{\partial \lambda_1} &= \mu_P - w_1\mu_1 - w_2\mu_2 = 0, \\ \frac{\partial L}{\partial \lambda_2} &= 1 - w_1 - w_2 = 0.\end{aligned}$$

Here in fact only the last two equations matter, as when we solve them we obtain unique values for w_1 and w_2 .

There is only one portfolio with the required return, and hence we cannot sensibly use the first two equations to find the “best” such portfolio.

To find the portfolio for the above case is Exercise Sheet 1, Question 5.

2.4 A general number of assets

We now consider a portfolio with n assets, and find the portfolio that minimises the variance σ_p^2 for a given expected return, μ_p .

It is possible that there are many such portfolios (or that there is no solution to the problem at all). To ensure that there is a unique solution, we shall assume;

- (i) asset returns are linearly independent, so no one can be expressed as a linear combination of any of the others,
- (ii) no two assets have the same expected return μ_i ,
- (iii) at least one asset has a return above the value of the expected return that we are seeking.

It is possible to demonstrate mathematically that the above conditions are necessary and sufficient for there to be such a unique portfolio, but we will not do this here.

To find the critical point, we must solve the following system of $n + 2$ linear equations (i.e. linear in the values to be determined w_i).

$$\frac{\partial L}{\partial w_1} = w_1 \sigma_1^2 + \sum_{j \neq 1}^n w_j \sigma_{1j} - \lambda_1 \mu_1 - \lambda_2 = 0,$$

$$\vdots$$

$$\frac{\partial L}{\partial w_i} = w_i \sigma_i^2 + \sum_{j \neq i}^n w_j \sigma_{ij} - \lambda_i \mu_i - \lambda_2 = 0,$$

$$\vdots$$

$$\frac{\partial L}{\partial w_n} = w_n \sigma_n^2 + \sum_{j \neq n}^n w_j \sigma_{nj} - \lambda_n \mu_n - \lambda_2 = 0,$$

$$\frac{\partial L}{\partial \lambda_1} = \mu_P - \sum_{i=1}^n w_i \mu_i = 0,$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - \sum_{i=1}^n w_i = 0.$$

2.5 Frontier portfolios

Using the methodology described above, if we are given a value of our desired expected return μ_P , we can find the optimal portfolio with minimum variance σ_P^2 .

If a different value of μ_P is chosen, then this will yield a different value of σ_P^2 .

Suppose that the asset with the smallest rate of return has rate $\mu_{\min}$ and the asset with the largest rate of return has rate $\mu_{\max}$.

Then we can find an optimal portfolio for all $\mu_{\min} \leq \mu_P \leq \mu_{\max}$.

We thus have a set of potentially optimal portfolios.

Some of these portfolios can be discounted, if another member of the set has both a higher expected rate of return and lower variance.

The remaining portfolios are called *frontier portfolios*.

Within this set of assets, the “best” portfolio must be one of the frontier portfolios.

FIGURE 1

3 The case with 3 assets

3.1 Finding the optimal portfolio

The three asset case is the simplest which has a non-trivial solution, i.e. there are many portfolios that give the required rate of return, but only one of them has minimum variance.

For three assets we must solve the simultaneous equations

$$\frac{\partial L}{\partial w_1} = w_1\sigma_1^2 + w_2\sigma_{12} + w_3\sigma_{13} - \lambda_1\mu_1 - \lambda_2 = 0, \quad (1)$$

$$\frac{\partial L}{\partial w_2} = w_2\sigma_2^2 + w_1\sigma_{12} + w_3\sigma_{23} - \lambda_1\mu_2 - \lambda_2 = 0, \quad (2)$$

$$\frac{\partial L}{\partial w_3} = w_3\sigma_3^2 + w_1\sigma_{13} + w_2\sigma_{23} - \lambda_1\mu_3 - \lambda_2 = 0, \quad (3)$$

$$\frac{\partial L}{\partial \lambda_1} = \mu_P - w_1\mu_1 - w_2\mu_2 - w_3\mu_3 = 0, \quad (4)$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - w_1 - w_2 - w_3 = 0. \quad (5)$$

We have five equations in five unknowns. We shall proceed to find the unique solution.

Taking equation (3) from equations (1) and (2) in turn gives

$$w_1(\sigma_1^2 - \sigma_{13}) + w_2(\sigma_{12} - \sigma_{23}) + w_3(\sigma_{13} - \sigma_3^2) - \lambda_1(\mu_1 - \mu_3) = 0 \quad (6)$$

and

$$w_2(\sigma_2^2 - \sigma_{23}) + w_1(\sigma_{12} - \sigma_{13}) + w_3(\sigma_{23} - \sigma_3^2) - \lambda_1(\mu_2 - \mu_3) = 0 \quad (7)$$

Taking $(\mu_1 - \mu_3)$ times equation (7) from $(\mu_2 - \mu_3)$ times equation (6) yields

$$\begin{aligned} & w_1[(\mu_2 - \mu_3)(\sigma_1^2 - \sigma_{13}) - (\mu_1 - \mu_3)(\sigma_{12} - \sigma_{13})] + \\ & w_2[(\mu_2 - \mu_3)(\sigma_{12} - \sigma_{23}) - (\mu_1 - \mu_3)(\sigma_2^2 - \sigma_{23})] + \\ & w_3[(\mu_2 - \mu_3)(\sigma_{13} - \sigma_3^2) - (\mu_1 - \mu_3)(\sigma_{23} - \sigma_3^2)] = 0 \Rightarrow \end{aligned}$$

$$\begin{aligned} & [\mu_2(\sigma_1^2 - \sigma_{13}) + \mu_1(\sigma_{13} - \sigma_{12}) + \mu_3(\sigma_{12} - \sigma_1^2)]w_1 + \\ & [\mu_3(\sigma_2^2 - \sigma_{12}) + \mu_2(\sigma_{12} - \sigma_{23}) + \mu_1(\sigma_{23} - \sigma_2^2)]w_2 + \\ & [\mu_1(\sigma_3^2 - \sigma_{23}) + \mu_3(\sigma_{23} - \sigma_{13}) + \mu_2(\sigma_{13} - \sigma_3^2)]w_3 = 0. \quad (8) \end{aligned}$$

We have now eliminated λ_1 and λ_2 from our equations.

The three remaining equations are (4), (5) and (8).

$$\mu_P - w_1\mu_1 - w_2\mu_2 - w_3\mu_3 = 0, \quad (4)$$

$$1 - w_1 - w_2 - w_3 = 0, \quad (5)$$

$$\begin{aligned} & [\mu_2(\sigma_1^2 - \sigma_{13}) + \mu_1(\sigma_{13} - \sigma_{12}) + \mu_3(\sigma_{12} - \sigma_1^2)]w_1 + \\ & [\mu_3(\sigma_2^2 - \sigma_{12}) + \mu_2(\sigma_{12} - \sigma_{23}) + \mu_1(\sigma_{23} - \sigma_2^2)]w_2 + \\ & [\mu_1(\sigma_3^2 - \sigma_{23}) + \mu_3(\sigma_{23} - \sigma_{13}) + \mu_2(\sigma_{13} - \sigma_3^2)]w_3 = 0. \end{aligned} \quad (8)$$

Equation (5) gives $w_3 = 1 - w_1 - w_2$ which substituting into (4) gives

$$\begin{aligned} w_1(\mu_1 - \mu_3) + w_2(\mu_2 - \mu_3) &= \mu_P - \mu_3 \Rightarrow \\ w_2 &= \frac{(\mu_P - \mu_3) - w_1(\mu_1 - \mu_3)}{\mu_2 - \mu_3}. \end{aligned} \quad (9)$$

Substituting (9) into (5) and rearranging gives

$$w_3 = \frac{(\mu_2 - \mu_P) + w_1(\mu_1 - \mu_2)}{\mu_2 - \mu_3}. \quad (10)$$

We could then substitute Equations (9) and (10) to get a unique (and long) expression for w_1 , which we could then use to find w_2 and w_3 .

One advantage of doing this is that we would then have a general expression for w_1, w_2 and w_3 for any set of portfolio parameters.

Instead we shall consider a numerical example. The form we have left the solution in proves a good one for solving particular numerical examples.

Example 2

Following Example 1, consider a portfolio with three assets S_1, S_2 and S_3 which have expected rates of return 0.1, 0.15 and 0.2 respectively, and variance-covariance matrix

$$V = \begin{pmatrix} 0.1 & 0.1 & -0.1 \\ 0.1 & 0.2 & 0.1 \\ -0.1 & 0.1 & 0.6 \end{pmatrix}.$$

a) Find the optimal portfolio for this example, where the desired rate of return is $\mu_P = 0.15$.

SOLUTION

b) Repeat the above for the rate of return $\mu_P = 0.16$.

SOLUTION

3.2 Frontier portfolios for three assets

If we substitute for w_2 and w_3 from (9) and (10) in Equation (8), the equation can be expressed in the following form

$$A_1 + A_2w_1 + A_3\mu_P = 0. \quad (11)$$

where A_1, A_2 and A_3 depend upon the constants in the mean vector and the variance-covariance matrix only.

Suppose that we are interested in finding the variance σ_P^2 of the optimal portfolio for a range of values of μ_P .

Equation (11) shows that w_1 is linear in μ_P , and so from (8) and (9) so are w_2 and w_3 .

Since

$$\sigma_P^2 = \sum_{i,j=1}^n w_i w_j \sigma_{ij}$$

it is clear that there is a quadratic relationship between σ_P^2 and μ_P of the form

$$\sigma_P^2 = B_2\mu_P^2 + B_1\mu_P + B_0.$$

This quadratic relationship also holds for any number of assets n (including $n = 2$, see Exercise Sheet 2, Question 3).

Example 3

Following Example 2, consider a portfolio with three assets S_1, S_2 and S_3 which have expected rates of return 0.1, 0.15 and 0.2 respectively, and variance-covariance matrix

$$V = \begin{pmatrix} 0.1 & 0.1 & -0.1 \\ 0.1 & 0.2 & 0.1 \\ -0.1 & 0.1 & 0.6 \end{pmatrix}.$$

a) What is the optimal portfolio when $\mu_P = 0.2$.

SOLUTION

b) Use this piece of information and the earlier optimal portfolios from Example 2 to find the quadratic equation relating σ_P^2 and μ_P .

SOLUTION

e) Hence draw a plot of the solutions, and identify the frontier portfolios.

SOLUTION

4 A risk free asset

4.1 Modifying the original problem

Usually investors are not restricted to investing a fixed amount into a collection of risky assets.

They also have the option of investing in a risk free asset (e.g. Government bonds, although how “risk-free” this really is is questionable).

Similarly they may be able to borrow money at some fixed interest rate.

We shall assume that, as before, our investor invests into one of n assets $\{S_1, S_2, \dots, S_n\}$ but that this time a risk free asset S_0 can also be invested in.

S_0 is assumed to have rate of return ρ_0 , (so the mean rate of return is ρ_0 and the variance is 0).

The weights of the other assets w_i are chosen as before, but this time the sum of the weights does not have to equal 1.

If $\sum_{i=1}^n w_i < 1$ then the remaining $1 - \sum_{i=1}^n w_i$ is invested in the risk free asset.

If $\sum_{i=1}^n w_i > 1$ then the extra $\sum_{i=1}^n w_i - 1$ is borrowed at interest rate ρ_0 .

Of course in reality, borrowing rates would probably be somewhat higher than investment rates.

The problem is now to minimise the risk, whilst achieving the return, with no restriction on the sum of the weights.

We wish to minimise

$$\frac{1}{2}f(w_1, \dots, w_n) = \sum_{i,j=1}^n w_i w_j \sigma_{ij},$$

subject to

$$g(w_1, \dots, w_n) = \sum_{i=1}^n w_i \mu_i + \rho_0(1 - \sum_{i=1}^n w_i) - \mu_{P+} = 0,$$

where μ_{P+} is the target rate of return, including investment in the risk free asset.

We thus consider the Lagrangian

$$L(w_1, \dots, w_n | \lambda_1, \lambda_2) = f(w_1, \dots, w_n) - \lambda g(w_1, \dots, w_n),$$

and differentiate with respect to $w_1, \dots, w_n$ and λ as before.

4.2 The case with two risky assets

We will consider the most simple non-trivial problem.

This is before where there are three assets, but here one of them is the risk free one, and so there are only two risky assets needed.

$$\frac{\partial L}{\partial w_1} = w_1\sigma_1^2 + w_2\sigma_{12} - \lambda(\mu_1 - \rho_0) = 0, \quad (12)$$

$$\frac{\partial L}{\partial w_2} = w_2\sigma_2^2 + w_1\sigma_{12} - \lambda(\mu_2 - \rho_0) = 0, \quad (13)$$

$$\frac{\partial L}{\partial \lambda} = -w_1\mu_1 - w_2\mu_2 - \rho_0(1 - w_1 - w_2) + \mu_{P+} = 0. \quad (14)$$

Equation (14) rearranges to

$$w_1(\rho_0 - \mu_1) + w_2(\rho_0 - \mu_2) = \rho_0 - \mu_{P+}. \quad (15)$$

Taking $(\mu_1 - \rho_0)$ times Equation (13) from $(\mu_2 - \rho_0)$ times Equation (12) gives

$$\begin{aligned} & w_1[(\mu_2 - \rho_0)\sigma_1^2 - (\mu_1 - \rho_0)\sigma_{12}] + \\ & w_2[(\mu_2 - \rho_0)\sigma_{12} - (\mu_1 - \rho_0)\sigma_2^2] = 0. \end{aligned} \quad (16)$$

Thus we have two equations in two unknowns w_1 and w_2 and we can solve to give the unique solution.

It is relatively straightforward to show that the optimal frontier when there is a risk free asset is just a linear combination of a single portfolio of risky assets (one which lies on the original frontier when there was no risk free asset) and the risk free asset.

Different weightings of the risky portfolio and the risk free asset give different levels of return.

This is illustrated in Figure 2 below.

FIGURE 2

The minimum expected level of return on the frontier is ρ_0 .

The maximum expected level of return is unbounded, provided our investor can borrow an unlimited amount to invest in the risky portfolio.

The corresponding variance associated with this return is also be unbounded, the higher the return, the higher the variance.

Example 4

Consider a portfolio with three assets S_0 , S_1 and S_2 where S_0 is a risk free asset with expected rate of return 0.05 and S_1 and S_2 have expected rates 0.1 and 0.15 respectively. The variance-covariance matrix for S_1 and S_2 is

$$V = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}.$$

Find the optimal portfolio for this example, where the desired rate of return is $\mu_{P+} = 0.1$.

SOLUTION

Note that you will build upon the work from the last two sections in the module “Managing Risk and Uncertainty” later in the course.