

Name: _____

Instructions

- Answer each question to the best of your ability.
- **Read each question carefully!**
- These problems are designed to be difficult and multi-faceted. If you do not know how to completely solve a problem, or you try but do not get the correct answer, you can still receive a large amount of partial credit. Tips for earning partial credit:
 - Show your work!
 - Find as much relevant information as you can, and label it clearly.
 - If you are making any assumptions in the problem, outline them clearly (preferably at the beginning of the problem).
 - If you can't solve the problem as presented, try solving a similar but simpler problem.
- **Unless the problem specifies otherwise** (e.g. asks for a decimal approximation):
 - use fractions (possibly improper ones) instead of mixed numbers or decimals;
 - leave non-perfect square roots and logarithms (like $\sqrt{2}$ or $\sqrt{7}$) as-is, but simplify as much as you can (so write $2\sqrt{7}$ instead of $\sqrt{28}$, but do NOT approximate to 5.3);
 - do not de-nest square roots (so $\sqrt{3 + \sqrt{7}}$ is fine);
 - leave π and e as they are;
 - leave i as-is if it is not being taken to a power (so $3i$ should be left alone, but $3i^3$ should be simplified to $-3i$).
- Due Date: Tuesday, November 6th, by the start of class. There will be NO late submissions accepted - any portions of the exam not completed and turned in by this date are counted as 0.

Formulas and definitions you may find useful

This is a take-home final, so no formula sheet will be provided. Your notes are your formula sheet!

1 Bicycle Race

You are riding your bicycle to school. Your (equally sized and perfectly circular) wheels have a radius of 1 foot from the center of the wheel to the outer edge that touches the road. You may assume no slippage - if the wheel rotates a full turn, then you move forward a distance equal to the circumference of the wheel.

For this problem, decimal approximations are acceptable.

1. If your wheels are spinning at a rate of 10 radians per second, how fast is the bike traveling in miles per hour?
2. You live exactly 10 miles away from school. If it takes you exactly 1 hour to make it to school, how fast on average are your wheels spinning in radians per second during the trip?
3. Right before you take off, you draw a red dot on the edge of your wheel, at the point where it touches the road. When you make it to school (10 miles away), you draw another dot at the new point where the wheel touches the road. Assuming the road to school is perfectly straight and level, what is the distance between these two dots?

2 We Will Rock You

You accidentally travel back in time to the year 1899. There are no computers, so out of boredom you decide to challenge Billy, one of the local kids, to a rock-throwing contest. Whoever can throw a rock the farthest wins!

You think you should have the edge, because you aren't malnourished, but the 12 hour days Billy spends swinging his pickaxe in the coal mines have given him some considerable arm muscle. To make sure you win, you'll need to employ your knowledge of trigonometry!

To simplify the calculations, you may assume no air resistance for this problem, so after the rock leaves your hand, the only thing affecting it is gravity. For this problem, decimal approximations are acceptable.

1. You can throw the rock at an initial speed of 20 meters per second at any angle you want, from an initial height of 1 meter above the ground. What is the maximum horizontal distance you can make the rock travel before it hits the ground? (*Hint: Use trigonometry to break up the speed into a horizontal component and a vertical component. The horizontal speed won't change as the rock travels, and the vertical speed will change according to gravity.*)
2. Billy can throw the rock at a speed of 25 meters per second. He reasons that if he throws the ball at a steeper angle, it will stay in the air longer, so he throws at an angle of 70 degrees from the horizontal. You feel bad about beating a 10 year old, but you're still competitive, so you decide to beat him by exactly 1 meter. At what angle (from the horizontal) should you throw the rock to accomplish this? (You may give your answer in either degrees or radians, just make it clear which one you are using.)

4 Somebody to Love

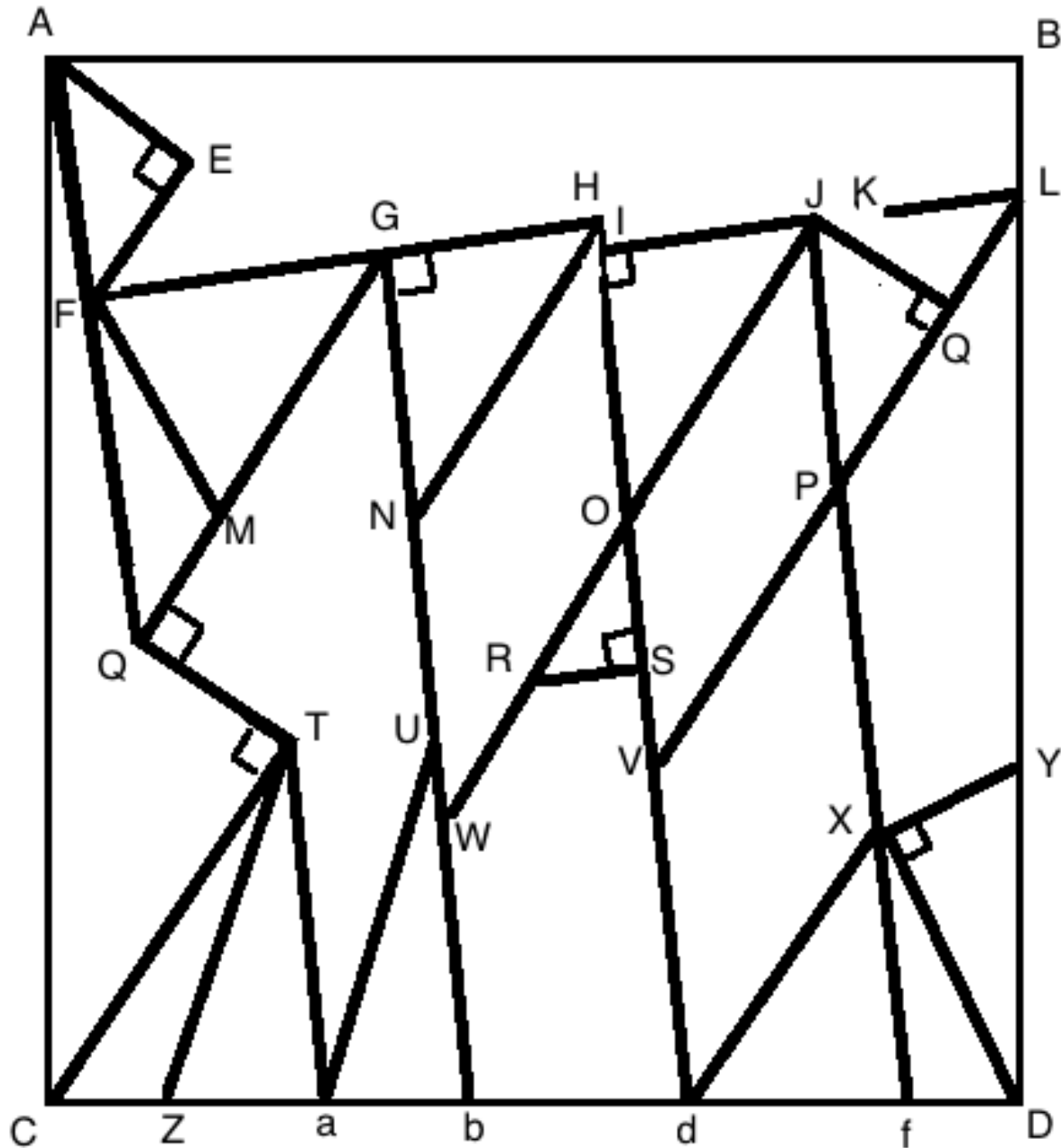
A species of lovebird has been observed to have very peculiar mating habits, which are followed precisely. These birds mate for life, and the female of each pair lays 2 eggs at a time, twice per year, on the autumn and spring equinoxes. Almost as if they were trying to make the lives of budding ornithologist-mathematicians easier, the eggs are always split evenly between male and female, all eggs hatch after precisely 30 days, and the offspring mature quickly so they are ready to lay eggs on the equinox immediately following their birth. A mating pair lays eggs for just one year (a total of 2 mating periods).

You are studying a population of these lovebirds. You counted yesterday, and the population contained 24 individuals, none of which have laid any eggs yet.

1. Assuming no birds die between now and then, how many total birds will be in the population on June 12th, 2021?
2. (Bonus) Assuming no birds die before they finish their mating cycle, how many birds will have been born by May 5th, 2050 (counting the original 24)?

5 Under Pressure

You are a graffiti artist. You have just finished the design of your magnum opus (the letters on the picture do not appear in the design, they are just to help with the math):



QUESTIONS AND INFO ARE ON THE NEXT PAGE

Important information:

- Decimal approximations are acceptable for this problem.
- You plan to draw this image on a 10 foot by 10 foot section of wall.
- Side AE has a length of 2 feet.
- Side FQ has a length of 3.5 feet.
- Side FG has the same length as side HN.
- Angle Wba has a measure of $\frac{5\pi}{12}$ radians.
- Angle MGN has a measure of $\frac{\pi}{4}$ radians.
- Lines AQ, aT, Gb, Hd, and Jf are all parallel to each other.
- Lines FE, QG, HN, JW, LV, and Xd are all parallel to each other.
- Lines Tz and Ua are parallel to each other.
- If you draw a line through points M and N, it will be a horizontal line.
- If you draw a line through points H and J, it will be a horizontal line.
- If you draw a line through points T and U, it will be a horizontal line.
- Points I, J, K, and L are co-linear (you can draw a straight line through all of them).
- Point R is halfway in between point O and point W.

1. How far away is Z from C?

2. How far away is Y from D?

3. What will be the perimeter of triangle ROS?

6 We Are the Champions

You are playing basketball with your friends, using an NBA regulation basketball and hoop. The score is 42-44, and your team is losing. With 10 seconds on the clock, you need to make the winning 3 point shot to win the game.

However, you don't just want to win, you want to win in *style*, so you decide to make the shot "nothing but net" (no backboard rebounding, no touching the rim).

Assumptions:

- At the moment it leaves your hand, the center of the ball is 5 feet above the ground and 24 feet from the center of the hoop.
 - The ceiling of the room is 20 feet high, and your shot does not count if it hits the ceiling.
 - The floor of the court is perfectly flat and level.
 - The basketball is perfectly spherical.
 - Acceleration due to gravity is 32 feet per second squared, downwards.
 - Air resistance is negligible (you can safely ignore it).
1. How fast should you throw the ball initially, and at what angle? Indicate your units clearly. (There are multiple possible answers here!)

7 Don't Stop Me Now

Consider the following theorem:

Theorem 1. *For every positive integer n , there is some perfect square in between n and $2n$ (inclusive).*

Also consider the following attempted proof that Theorem 1 is true:

Here is a table for n from 1 to 10:

n	$2n$	perfect squares in between
1	2	1
2	4	4
3	6	4
4	8	4
5	10	9
6	12	9
7	14	9
8	16	9, 16
9	18	9, 16
10	20	16

These values all satisfy the theorem, so it will always be true no matter what n is.

1. This “proof” does not actually prove the theorem. Explain why it does not.

Now, consider the following proof that Theorem 1 is false:

The “Attempted Proof that Theorem 1 is true” above was insufficient to prove Theorem 1. Therefore, Theorem 1 is false.

2. Does this prove that Theorem 1 is false? Explain your answer.

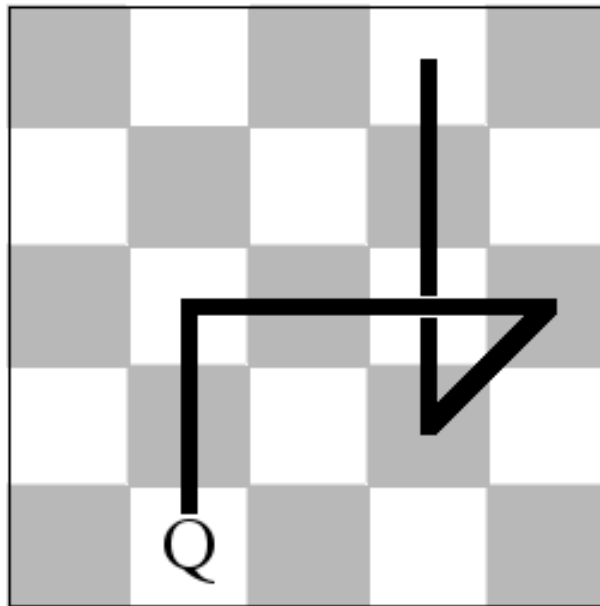
8 Killer Queen

In chess, the queen is often considered the most powerful piece. This is because it can move any distance in any one of 8 different directions: up, down, left, right, and any of the 4 diagonals.

Some definitions we will use in this problem:

- A *move* is a direction plus a distance, where the possible directions are abbreviated as U, D, L, R, UL, UR, DL, and DR. “R5”, “D3” and “UL7” are all examples of moves. Any non-negative integer is a valid distance, including 0 (a distance of 0 means the queen does not move).
- A *sequence of moves* is an ordered list of moves, written in square brackets. For example, the sequence of moves [D3UL5R1] corresponds to “move down 3 spaces, then move to the upper left 5 spaces diagonally, then move right 1 space”.
- Two sequences of moves are *equivalent* if the queen will end up in the same position after performing either sequence. We denote equivalence with the = symbol. For example, the sequence [D3UL5R1] = [U2L4], because moving left 5 and then right 1 is the same as just moving left 4.
- The *composition* of two sequences of moves is the sequence of moves that results from performing the first sequence of moves, and then the second sequence. We will denote composition of sequences with the $\oplus$ symbol. For example, [R3D4] $\oplus$ [L2UL1] = [R3D4L2UL1].

Here is an illustration of the move sequence [U2R3DL1U3]:



Traditionally, the game of chess is played on an 8x8 board, and involves two teams of 16 pieces each. However, in this problem, we will not be so restricted - there will be just 1 queen, and it will be on an infinite chess board.

QUESTIONS ARE ON THE NEXT PAGE

1. Is the $\oplus$ operator associative with respect to equivalence? In other words, if x , y , and z are sequences of moves, does $x \oplus (y \oplus z) = (x \oplus y) \oplus z$?
2. Is the $\oplus$ operator commutative with respect to equivalence? In other words, if x and y are sequences of moves, does $x \oplus y = y \oplus x$?
3. For each of the following (infinite) sets, determine whether they are closed¹ under the $\oplus$ operator. Justify your answers! You may assume that the integers are closed under addition and subtraction.
 - (a) The set $\mathbb{SM}$ of all possible finite sequences of moves.
 - (b) The set $\mathbb{SM}_1$ of all possible sequences that contain exactly 1 move.
 - (c) The set $\mathbb{SM}_2$ of all possible sequences that contain exactly 2 moves.
 - (d) The set $\mathbb{UM}$ of all possible sequences that contain only U moves. (UL and UR moves are not permitted in this or the next question.)
 - (e) The set $\mathbb{UM}_1$ of all possible sequences that contain exactly 1 U move.

¹If x and y are moves, and $x \oplus y$ is equivalent to some element of the set, that counts as $x + y$ being in the set. This is similar to how $\frac{3.4}{6.8}$ is in the set of rational numbers $\mathbb{Q}$ even though it doesn't directly fit the "p/q for integers p and q" format, because it is equivalent to the fraction $\frac{1}{2}$ which does fit the format.

9 Another One Bites the Dust

The *norm* of a complex number $a + bi$ ($a, b \in \mathbb{R}$) is defined to be $\sqrt{a^2 + b^2}$.

For this problem, we will use C_1 to denote the set of all complex numbers that have a norm of exactly 1. Examples of complex numbers in this set include 1, i , -1 , and $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)$.

1. Is C_1 closed under (justify your answers):

(a) Addition

(b) Subtraction

(c) Multiplication

(d) Division

2. Complex numbers can be graphed on something called the Complex Plane, with the x-coordinate being the real part and the y-coordinate being the imaginary part. For example, the complex number $3 + 5i$ would be graphed at the point (3,5).

If you graph every point in C_1 in this way, what shape will be formed?

10 Play the Game

Sekora International Casino wants to introduce a new dice game that uses fair 6-sided dice, and works as follows:

First, you roll 2 dice, and take the difference of the rolls. Call this difference x . Then, you roll x dice, and add the totals of those rolls together. This new total is your payout in dollars.

For example, if the first 2 dice come up 2 and 6, the difference is 4. So, you would then roll 4 dice. Say those 4 dice come up 3,1,1,2, then your payout is $3+1+1+2 = 7$ dollars.

You have been tasked with setting the price for one play of this game.

1. If Sekora International Casino wants to make an average of \$1 in profit per play, how much should we charge for one play?