

Question #9

a) $\int_1^2 \frac{1}{\sqrt{x}} dx$, $\Delta x = \frac{1}{\sqrt{x}}$

First we partition interval $[1, 2]$ into n subinterval

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}$$

Length of each of the interval is $k\Delta x = \frac{k}{n}$, $k=1, 2, \dots, n$

Right end point each of the interval $C_k = 1 + \frac{k}{n}$, $k=1, 2, \dots, n$

So, the Riemann sum approximate to the integral

$\int_1^2 \frac{1}{\sqrt{x}} dx$ with n interval is

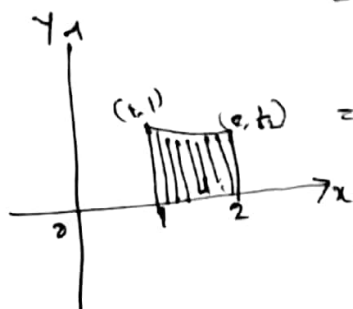
$$\begin{aligned} S_n &= \sum_{k=1}^n f(x_k) \Delta x \\ &= \sum_{k=1}^n \frac{1}{\sqrt{1+k/n}} \cdot \frac{1}{n} \\ &= \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1+k/n}} \end{aligned}$$

b) If $n=10$

then $C_k = 1 + \frac{k}{10}$, $k=1, 2, \dots, 10$

Then, the Riemann sum for $n=10$ interval is

$$= \frac{1}{10} \sum_{k=1}^{10} \frac{1}{\sqrt{1+k/10}}$$



$$= \frac{1}{10} \left(\frac{1}{\sqrt{1+\frac{1}{10}}} + \frac{1}{\sqrt{1+\frac{2}{10}}} + \frac{1}{\sqrt{1+\frac{3}{10}}} + \dots + \frac{1}{\sqrt{1+\frac{10}{10}}} \right)$$

$$= \frac{1}{\sqrt{10}} \left(\frac{1}{\sqrt{11}} + \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{13}} + \dots + \frac{1}{\sqrt{20}} \right)$$

$$= 6.814$$

This value an under estimation of the integral.

Since, Right end point of the interval gives is minimum value of $f(x)$ as f is a decreasing function.

So, Sum approximation to the integral using

right end point $L(P, f)$ and $L(P, f) \leq \int_1^2 \frac{1}{\sqrt{x}} dx$

Question #9

c) (i) f is monotone on $[1, 2]$ so f is integrable.

(ii) f has ~~an~~ antiderivative F is $f'(x) = \frac{1}{\sqrt{x}}$

$$f(x) = 2\sqrt{x} \text{ such that } F'(x) = f(x)$$

By Fundamental theorem of integral calculus,

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{x}} dx &= F(2) - F(1) \\ &= 2\sqrt{2} - 2\sqrt{1} \\ &= 2(\sqrt{2} - 1) \\ &\approx 0.828.\end{aligned}$$

d) If, take limit n tends to ∞

$$\text{Then } \lim_{n \rightarrow \infty} S_n = \int_1^2 \frac{1}{\sqrt{x}} dx = 2(\sqrt{2} - 1)$$

$$\text{ie } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1+k/n}} = 2(\sqrt{2} - 1)$$

$$e) \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{\sqrt{1+\frac{1}{n}}} + \frac{1}{\sqrt{1+\frac{2}{n}}} + \dots + \frac{1}{\sqrt{1+\frac{k}{n}}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1+\frac{k}{n}}}$$

$$= \underline{\underline{2(\sqrt{2} - 1)}}$$