Question 1

$$\int_0^1 \frac{\ln(x)}{\sqrt{x}}$$

Solve for:

$$\int \frac{\ln(x)}{\sqrt{x}}\, dx$$

Applying integration by parts:

$f = \ln(x)$; $f' = \dfrac{1}{x}$

$g' = \dfrac{1}{\sqrt{x}}$ $\quad g = 2\sqrt{x}$

$$\int fg' = fg - \int f'g$$

$$\Rightarrow 2\sqrt{x}\ln(x) - \int \frac{2}{\sqrt{x}}\, dx$$

$$= 2\sqrt{x}\ln(x) - 2\int \frac{1}{\sqrt{x}}\, dx \Rightarrow \int \frac{1}{\sqrt{x}} = 2\sqrt{x}$$

$$= 2\sqrt{x}\ln(x) - 4\sqrt{x} + c \Big|_0^1$$

$$= 2\ln(1) - 4$$

$$= -4$$

Question 2

$$x = \frac{y^2}{4} - \frac{1}{4} \qquad 1 \leq x \leq 5$$

(i) Definite integral in terms of x:

$$S = 2\pi \int_a^b y \sqrt{(f'(y))^2 + 1}\, dy$$

$$x = \frac{y^2}{4} - \frac{1}{4} \;; \quad y = \sqrt{4x+1}$$

$$f'(x) = \frac{2}{\sqrt{4x+1}}$$

$$S = 2\pi \int_1^5 \sqrt{4x+1} \cdot \sqrt{\left(\frac{2}{\sqrt{4x+1}}\right)^2 + 1}\, dx \quad = 2\pi \int_1^5 \sqrt{\cancel{4x+1} \cdot \frac{4}{\cancel{4x+1}} + 4x+1}$$

$$= 2\pi \int_1^5 \sqrt{4x+5}\, dx$$

Applying u-substitution: $u = 4x+5$, $\frac{du}{dx} = 4$; $dx = \frac{du}{4}$

$$= 2\pi \int_1^5 \sqrt{u}\, \frac{du}{4} = \frac{1}{2}\pi \int_1^5 \sqrt{u}\, du \;; \quad \int \sqrt{u}\, du = \frac{2u^{3/2}}{3}$$

$$= \frac{1}{3}\pi \cdot u^{3/2}$$

Substituting back

$$= \frac{\pi}{3}(4x+5)^{3/2} \Big|_1^5 = \frac{\pi}{3}(20+5)^{3/2} - 9^{3/2} = \frac{\pi}{3}(125-27)$$

$$= \frac{98\pi}{3} \text{ sq. units}$$

(ii) Definite integral in terms of y

$x = \frac{y^2}{4} - \frac{1}{4}$; $\frac{dx}{dy} = \frac{1}{2} y$

$y_a = \sqrt{4 \times 1 + 1} = \sqrt{5}$

$y_b = \sqrt{4 \times 5 + 1} = \sqrt{21}$

$$S = \int_{\sqrt{5}}^{\sqrt{21}} 2\pi y \sqrt{1 + \left(\frac{y}{2}\right)^2}\, dy = 2\pi \int_{\sqrt{5}}^{\sqrt{21}} \sqrt{y^2 \cdot 1 + \frac{y^4}{4}}\, dy$$

$$\Rightarrow 2\pi \int_{\sqrt{5}}^{\sqrt{21}} y \cdot \sqrt{\frac{4 + y^2}{4}} = \pi \int_{\sqrt{5}}^{\sqrt{21}} y \sqrt{4 + y^2}\, dy$$

Applying u-substitution; $u = y^2 + 4$, $\frac{du}{dy} = 2y$; $dy = \frac{du}{2y}$

$$= \frac{\pi}{2} \int_{\sqrt{5}}^{\sqrt{21}} \sqrt{u}\, du \quad ; \quad \int \sqrt{u}\, du = \frac{2}{3} u^{3/2}$$

$$= \frac{\pi}{3} u^{3/2} + C$$

Substituting back:

$$= \frac{\pi}{3} (y^2 + 4)^{3/2} \Big|_{\sqrt{5}}^{\sqrt{21}}$$

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$$= \frac{\pi}{3}(21+4)^{3/2} - \frac{\pi}{3}(5+4)^{3/2}$$

$$= \pi/_3 \,(125-27)$$

$$= \frac{98\pi}{3} \text{ sq. units}$$

Question 3

$$\int \frac{4x^4+14x^2+2}{4x^4+x^2}\,dx$$

Performing polynomial long division:

$$\begin{array}{r|l} & \quad 1 \\ 4x^4+x^2 & 4x^4+0x^3+14x^2+0x+2 \\ & \underline{4x^4+0x^3+x^2+0x+0} \quad - \\ & \qquad\qquad 13x^2+0x+2 \end{array}$$

$$= 1 + \frac{13x^2+2}{4x^4+x^2}$$

Take $\dfrac{13x^2+2}{4x^4+x^2} = \dfrac{13x^2+2}{x^2(4x^2+1)}$

$$\frac{13x^2+2}{x^2(4x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{4x^2+1}$$

$$= \frac{x^2(Cx+D) + x(4x^2+1)A + (4x^2+1)B}{x^2(4x^2+1)}$$

$$13x^2+2 = x^2(Cx+D) + x(4x^2+1)A + (4x^2+1)B$$

$$13x^2+2 = 4x^3A + x^3C + 4x^2B + x^2D + xA + B$$

$4A + C = 0$

$4B + D = 13$; $C = 0$ and $D = 5$

$B = 2$

$A = 0$

Thus: $\dfrac{4x^4 + 14x^2 + 2}{4x^4 + x^2} = 1 + \dfrac{2}{x^2} + \dfrac{5}{4x^2+1}$

$\Rightarrow \displaystyle\int dx + 2\int \frac{1}{x^2}\,dx + 5\int \frac{1}{4x^2+1}\,dx$

$\displaystyle\int dx = x$

$\displaystyle\int \frac{1}{x^2} = \frac{x^{-2+1}}{-1} = \frac{-1}{x}$

$\displaystyle\int \frac{1}{4x^2+1}\,dx \Rightarrow$ Applying U-substitution ; $U = 2x$

$\dfrac{du}{dx} = 2$; $dx = \dfrac{du}{2}$

$$= \int \frac{1}{u^2+1}\,\frac{du}{2} = \frac{1}{2}\int \frac{1}{u^2+1}$$

but $\displaystyle\int \frac{1}{u^2+1}\,du = \int \arctan(u)$

$$= \frac{1}{2}\arctan(2x)$$

$\Rightarrow x - \dfrac{2}{x} + \dfrac{5}{2}\arctan(x) + C$

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Question 4

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt \qquad y = f(x) \quad ; 9x^2 = y(y-3)^2$$

Find $\dfrac{ds}{dx}$

$$F(x) = \int_{\phi(x)}^{\psi(x)} f(t)\,dt + c$$

$$f'(x) = f(\psi(x)) \cdot \psi'(x) - f(\phi(x)) \cdot \phi'(x)$$

$$\frac{ds}{dx} = \sqrt{1+(f'(x))^2}\,\frac{d}{dx}\cdot x - \sqrt{1+(f'(a))^2}\,\frac{d}{dx}a$$

$$= \sqrt{1-(f(x))^2}$$

$$9x^2 = y(y-3)^2$$

$$18x = y'(y-3)[3y-3]$$

$$y' = \frac{18x}{3(y-3)(y-1)}$$

$$\frac{ds}{dx} = \sqrt{1+\left[\frac{18x}{3(y-3)(y-1)}\right]^2}$$

$$= \sqrt{1+\left[\frac{6x}{(y-3)(y-1)}\right]^2}$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{6x}{(f(x)-3)(f(x)-1)}\right)^2} \quad ; \quad f(x) \neq 3,\ f(x) \neq 1$$

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Question 5

$$\int \frac{2x}{x^2-x-2}\,dx$$

Factor the denominator: $x^2 - x - 2 = (x-2)(x+1)$

$$\Rightarrow \frac{2x}{(x+1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$= \frac{(x-2)A + (x+1)B}{(x-2)(x+1)}$$

$$2x = (x-2)A + (x+1)B$$
$$= xA + xB - 2A + B$$
$$= x(A+B) - 2A + B$$

$$A + B = 2 \qquad ; \; A = 2/3 \text{ and } B = 4/3$$
$$-2A + B = 0$$

$$= \frac{2/3}{x+1} + \frac{4/3}{x-2} \Rightarrow \frac{2}{3(x+1)} + \frac{4}{3(x-2)}.$$

$$= \frac{2}{3}\int \frac{1}{x+1}\,dx + \frac{4}{3}\int \frac{1}{x-2}\,dx \qquad ; \; \int \frac{1}{x+a}\,dx = \ln|x+a|$$

$$= \frac{2}{3}\ln|x+1| + \frac{4}{3}\ln|x-2| + C$$

Question 6

$$\int \frac{a}{a^2+x^2}\,dx$$

$x = a\tan\theta$

$\frac{dx}{d\theta} = a\sec^2\theta \quad dx = a\sec^2\theta\, d\theta$

$$\int \frac{a}{a^2 + a\tan^2\theta} \cdot a\sec^2\theta\, d\theta$$

$$= \int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \int d\theta$$

$$= \theta + C$$

$\theta = \tan^{-1}\left(\frac{x}{a}\right)$

$$\int \frac{a}{a^2+x^2} = \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} = \left[\tan^{-1}\left(\frac{+\infty}{a}\right) - \tan^{-1}\left(\frac{-\infty}{a}\right)\right]$$

$= \tan^{-1}(\infty) - \tan^{-1}(-\infty)$; $\tan^{-1}(\infty) = \pi/2$

$\tan^{-1}(-\infty) = -\pi/2$

$$= \frac{\pi}{2} + \pi/2$$

$$= \cancel{\emptyset} \quad \pi$$

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Question 7

$f(x) = 1/x$

$f'(x) = -1/x^2$

$$S = \int 2\pi f(x)\sqrt{1+(f'(x))^2}\, dx$$

$$= \int_1^\infty \frac{2\pi}{x}\sqrt{\frac{x^4+1}{x^4}}\, dx$$

$$= \int_1^\infty \frac{2\pi\sqrt{x^4+1}}{x^3}\, dx$$

Let $g(x) = 1/x$

$$\frac{1}{x} < \frac{\sqrt{x^4+1}}{x^3}$$

$$\int \frac{1}{x}\, dx < \int \frac{\sqrt{x^4+1}}{x^3}\, dx$$

$$2\pi\int_1^\infty \frac{1}{x}\, dx < 2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$$

$$= 2\pi(\ln(x))\Big|_1^\infty < 2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$$

$$\infty < 2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$$

Thus the surface area will be ∞

Question 9

$g(x) = 1$
$f(x) = 0$

Arc length of $g(x)$ for $x \in [a, b)$:

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_a^b \sqrt{1+0}\, dx$$

$$= \int_a^b dx$$

$$[x]_a^b = b - a \quad \ldots\ldots (i)$$

Area between $g(x)$ and $f(x)$, $x \in [a, b]$

$$= \int_a^b (g(x) - f(x))\, dx = \int_a^b (1-0)\, dx \quad ; g(x) = 1, f(x) = 0$$

$$\int_a^b dx = [x]_a^b = b - a \quad \ldots\ldots (ii)$$

Thus, arclength of $g(x)$ and Area between $g(x)$ and $f(x)$, $x \in [a, b]$ are same.

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Question 9

$$h(x) = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$$

Length from $x = 1$ to $x = 3$

$$L = \int_a^b \sqrt{(f'(x))^2 + 1}\, dx$$

$$f'(x) = \left(\frac{x^3}{6} + \frac{1}{2x}\right)' = \frac{x^4 - 1}{2x^2} \quad \text{ie} \quad \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

$$L = \int_1^3 \sqrt{\left(\frac{x^4-1}{2x^2}\right)^2 + 1}\, dx = \int_1^3 \frac{\sqrt{(x^4+1)^2}}{2x^2}\, dx$$

$$= \frac{1}{2}\int_1^3 \frac{\sqrt{(x^4+1)^2}}{x^2}\, dx = \frac{1}{2}\int_1^3 (x^2 + x^{-2})\, dx$$

$$\frac{1}{2}\left[\frac{x^3}{3} + \frac{x^{-1}}{-1}\right]_1^3$$

$$= \frac{1}{2}\left[\frac{27}{3} - \frac{1}{3} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[9 - \frac{2}{3} + 1\right]$$

$$= \frac{14}{3}$$