Question 1.

$$\int e^{\cos(x)} \sin(2x)\,dx$$

$\sin(2x) = 2\sin(x)\cos(x)$

$$= 2\int e^{\cos(x)} \sin(x) \cos(x)$$

Let $u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$; $dx = \frac{-du}{\sin(x)}$

$$\Rightarrow -2\int e^{u} \cancel{\sin(x)} \times u \frac{du}{\cancel{\sin(x)}}$$

$$= -2\int ue^{u}$$

Integrating by parts: $\int fg' = fg - \int f'g$

$$\int ue^{u}\,du = ue^{u} - \int e^{u}\,du$$

Taking $\int e^{u}\,du \Rightarrow \frac{e^{u}}{\ln(e)} = e^{u}$

Therefore, $\int ue^{u}\,du = ue^{u} - e^{u}$

Plugging in -2 :

$$-2(ue^{u} - e^{u}) = 2e^{u} - 2e^{u}u$$

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but $u = \cos(x)$

$$= 2e^{\cos(x)} - 2e^{\cos(x)}\cos(x) + C$$

Verifying answer by differentiation:

$$\frac{d}{dx} 2e^{\cos(x)} = -2\sin(x)\, e^{\cos(x)}$$

$$\frac{d}{dx} 2e^{\cos(x)}\cos(x) = -2\sin(x)\, e^{\cos(x)}\cos(x) + 2e^{\cos(x)}(-\sin(x))$$

$$\Rightarrow -2\sin(x)\, e^{\cos(x)} + 2\sin(x)\, e^{\cos(x)}\cos(x) + 2e^{\cos(x)}(\sin(x))$$

$$= 2\sin(x)\, e^{\cos(x)}\cos(x)$$

Note: $\sin(2x) = 2\sin(x)\cos(x)$

$$= e^{\cos(x)}\sin(2x) + C$$

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Question 2

$$\int_0^4 h''(\sqrt{y})\,dy \quad \text{if } h(0)=1,\ h(2)=3,\ h'(2)=4$$

$$\Rightarrow I = \int_0^4 h''(\sqrt{y})\,dy \quad \ldots\ldots(i)$$

Substituting $\sqrt{y} = t$

$$\frac{1}{2\sqrt{y}} = dt$$

$$dy = 2\sqrt{y}\,dt \quad ; \quad dy = 2t\,dt$$

Lower limit : $\sqrt{0} = 0$

Upper limit : $\sqrt{4} = 2$

From (i) ;

$$I = \int_0^2 h''(t) \cdot 2t\,dt$$

$$I = 2\int_0^2 t \cdot h''(t)\,dt \quad \ldots\ldots(ii)$$

$$\frac{d}{dt}(t \times h') = th'' + h'$$

From (ii) ; $$I = 2\int_0^2 \left[\frac{d}{dt}(t \cdot h'(t)) - h'(t)\right] dt$$

$$I = 2\int_0^2 \frac{d}{dt}(t \cdot h'(t))\,dt - 2\int_0^2 h'(t)\,dt \quad \ldots\ldots(iii)$$

From (iii) ;

$$I = 2\left[t\,h'(t)\right]_0^2 - 2\left[h(t)\right]_0^2$$

$$= 2\left[2 \cdot h'(2) - 0\right] - 2\left[h(2) - h(0)\right]$$

$$= 2(2 \cdot 4 - 0) - 2(3 - 1)$$

$$= 16 - 4$$

$$= 12$$

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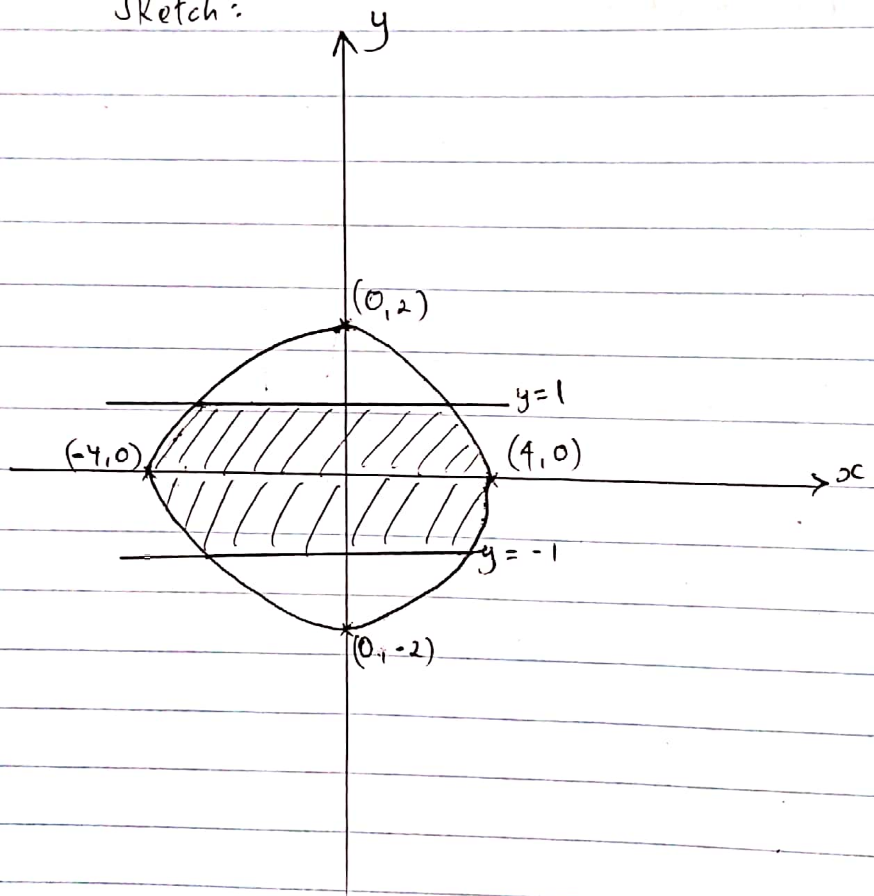
Question 3

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$\Rightarrow$ Find area of region inside the eclipse between $y=-1$ and $y=1$

Sketch:



Required area: $4\int_0^1 x\,dy$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\frac{x^2}{16} = 1 - \frac{y^2}{4}$$

$$x = 4\sqrt{1 - \frac{y^2}{4}}$$

$$\cancel{x} = 2\sqrt{4-y^2}$$

$$= 4\int_0^1 2\sqrt{4-y^2}\,dy$$

$$= 8\int \sqrt{4-y^2}\, dy$$

Applying trigonometric substitution:

$$y = 2\sin(u) \qquad dy = \cos(u)\,du$$
$$u = \sin^{-1}\left(\frac{y}{2}\right)$$

but $1-\sin^2(u) = \cos^2(u)$

$$= \int 2\cos(u)\sqrt{4-4\sin^2(u)}\,du = \int 2\cos(u)\; 2\sqrt{1-\sin^2 u}$$

$$= 4\int \cos^2(u)\,du$$

Applying reduction formula: $\int \cos^n(u)\,du = \frac{n-1}{n}\int \cos^{n-2}(u)\,du + \frac{\cos^{n-1}(u)\sin(u)}{n}$

$$\Rightarrow \frac{\cos(u)\sin(u)}{2} + \frac{1}{2}\int du$$

$$= 4\left[\frac{\cos(u)\sin(u)}{2} + \frac{u}{2}\right] \quad ; \text{Replacing } u \text{ with } \sin^{-1}\left(\frac{y}{2}\right)$$

$$= 4\left[\frac{\cos(\sin^{-1} y/2)\sin(\sin^{-1}(y/2))}{2} + \frac{1}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]$$

$$= 4\left[\frac{\sqrt{1-y^2/4}\cdot y/2}{2} + \frac{1}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]$$

$$= \left[y\sqrt{1-\frac{y^2}{4}} + 2\sin^{-1}\left(\frac{y}{2}\right)\right]_0^1$$

$$= \left[\frac{\sqrt{3}}{2} + \pi/3\right] - 0 \qquad = \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) \text{ square units}$$

Question 4

$$\int \tan^5(x) \cos^2(x)\, dx$$

$$\cos(x) = \frac{1}{\sec(x)}$$

$$\sec^2(x) = \tan^2(x) + 1$$

$$\Rightarrow \int \sec^2(x) \frac{\tan^5(x)}{(\tan^2(x)+1)^2}\, dx$$

Substituting $u = \tan(x)$

$$\frac{du}{dx} = \sec^2(x)$$

$$dx = \frac{du}{\sec^2(x)}$$

$$= \int \cancel{\sec^2(x)} \frac{u^5}{(u^2+1)^2} \frac{du}{\cancel{\sec^2(x)}}$$

$$= \int \frac{u^5}{(u^2+1)^2}\, du$$

Substitute $v = u^2 + 1$; $\frac{dv}{du} = 2u$, $du = \frac{1}{2u} dv$

$u = \sqrt{v-1}$

$$= \int \frac{\cancel{u}(v-1)^2}{v^2} \frac{dv}{2\cancel{u}} = \frac{1}{2} \int \frac{(v-1)^2}{v^2}\, dv$$

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Taking $\frac{(v-1)^2}{v^2} dv \Rightarrow \frac{v^2+1-2v}{v^2}$

$$= \frac{v^2}{v^2} + \frac{1}{v^2} - \frac{2}{v}$$

$$\frac{1}{2}\int\left(\frac{v^2}{v^2} + \frac{1}{v^2} - \frac{2}{v}\right) dv \Rightarrow \int \frac{v^2}{v^2} = 1$$

$$= -1\int \frac{1}{v} dv + \frac{1}{2}\int \frac{1}{v^2} dv + \frac{1}{2}\int dv$$

$$= -\ln(v) + \frac{v}{2} - \frac{1}{2v}$$; but $v = u^2+1$

$$= -\ln(u^2+1) + \frac{u^2+1}{2} - \frac{1}{2(u^2+1)}$$; Also, $u = \tan(x)$

$$= -\ln(\tan^2(x)+1) - \frac{\tan^2 x+1}{2} - \frac{1}{2(\tan^2(x)+1)}$$

$\tan^2(x) + 1 = \sec^2(x)$

$$= -\ln(\sec^2(x)) - \frac{\sec^2(x)}{2} - \frac{1}{2(\sec^2(x))} + C$$

Question 5

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx \quad \text{for} \quad x > 2/5$$

Let $x = \frac{2}{5}\sec(u)$ $\qquad u = \sec^{-1}(5x/2)$

$dx = 2/5 \tan(u)\sec(u)\,du$

$$= \int \frac{\sqrt{25 \times 4/25 \sec^2(u) - 4}}{2/5\sec(u)} \times 2/5 \tan(u)\sec(u)\,du$$

$$= \int \frac{2\sqrt{\sec^2(u)-1}}{2/5\,\cancel{\sec}(u)} \times 2/5\tan(u)\,\cancel{\sec}(u)\,du$$

But $\tan^2(u) = \sec^2(u) - 1$

$$= \int 2\tan(u)\tan(u)\,du$$

$$= 2\int \tan^2(u)\,du$$

$$= 2\int -1 + \sec^2(u)\,du$$

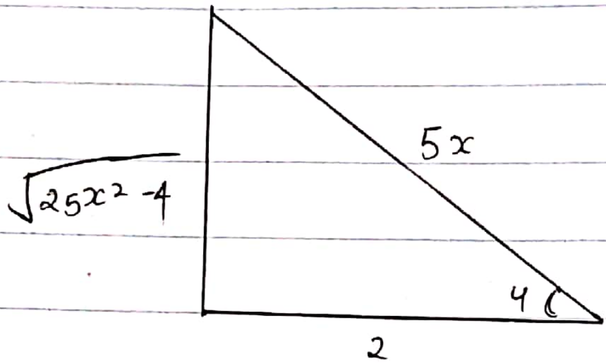
$$= -2\int du + 2\int \sec^2(u)\,du \quad \Rightarrow \quad \int \sec^2(u)\,du = \tan(u)$$

$$= -2u + 2\tan(u)$$

Substituting back $u = \sec^{-1}(5x/2)$

$$= -2\sec^{-1}\left(\frac{5x}{2}\right) + 2\tan\left(\operatorname{arcsec}\left(\frac{5x}{2}\right)\right)$$

Solve for $\tan\left(\sec^{-1}\left(5x/2\right)\right)$



$$\tan u = \frac{\sqrt{25x^2-4}}{2}$$

$$= -2\sec^{-1}\left(\frac{5x}{2}\right) + \sqrt{25x^2-4} + C$$

Quastion 6

$$\int_a^b p(x)\, q''(x)\, dx \qquad \begin{array}{l} p(a) = p(b) = 0 \\ q(a) = q(b) = 0 \end{array}$$

Applying product rule: of integration:

$$\int p(x)\, q''(x)\, dx = p(x) \int q(x)\, dx - \int p(x)\, dx \int q''(x)\, dx$$

$$= p(x) \int_a^b q''(x)\, dx - \int_a^b \frac{d}{dx} p(x)\, dx \cdot \int_a^b q''(x)\, dx$$

$$= p(x) \cdot q'(x) \Big|_a^b - \int_a^b p'(x)\, q'(x)\, dx$$

$$= p(b) \cdot q'(b) - p(a)\, q'(a) - \int_a^b p'(x)\, q'(x)\, dx$$

$$\Rightarrow p(b) \cdot q'(b) - p(a)\, q'(a) = 0$$

$$= - \int_a^b p'(x)\, q(x)\, dx$$

$$= - \left[p'(x) \cdot \int_a^b q'(x)\, dx - \int_a^b \left(\frac{d}{dx} p'(x)\, dx \int q'(x)\, dx \right) dx \right.$$

$$= - \left[\left(p'(x)\, q(q) \right)_a^b \right] - \left[\int_a^b p''(x) \cdot q(x)\, dx \right]$$

$$= - \left[p'(b) \cdot q(b) - p'(a)\, q(a) \right] + \left[\int_a^b p''(x)\, q(x)\, dx \right.$$

$$= -\left[0\right] + \int_a^b p''(x)\, q(x)\, dx$$

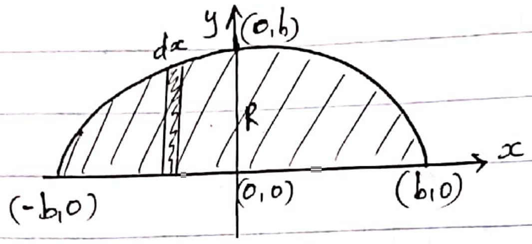
$$= \int_a^b p''(x)\, q(x)\, dx$$

Thus $\displaystyle\int_a^b p(x)\, q''(x)\, dx = \int_a^b p''(x)\, q(x)\, dx$

Question 7

~~Radius = 0.5 π b²~~ Radius = b

Area = $0.5\pi b^2$



$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = b^2$$

$$y = \sqrt{b^2 - x^2}$$
$$dA = \sqrt{b^2 - x^2}\, dx$$

$$A = \int_{-b}^{b} \sqrt{b^2 - x^2}\, dx = 2\int_{0}^{b} \sqrt{b^2 - x^2}\, dx$$

Applying trigonometric substitution.

$$x = b\sin(u) \;;\; u = \sin^{-1}\left(\frac{x}{b}\right) \;;\; dx = b\cos(u)\, du$$

$$= \int b\cos(u)\, b\sqrt{1 - \sin^2 u}\; du$$

but $1 - \sin^2 u = \cos^2 u$

$$= \int b^2 \cos^2(u)\, du = b^2 \int \cos^2(u)\, du$$

Applying reduction formula:

$$b^2 \int \cos^2(u)\, du = b^2\left[\frac{\cos(u)\sin(u)}{2} + \frac{1}{2}\int du\right]$$

$$= \frac{\cos(u)\ \sin(u)}{2} + \frac{u}{2} \quad ; \quad \text{but } u = \arcsin\left(\frac{x}{b}\right)$$

$$= b^2\left[\frac{\cos\left(\arcsin \frac{x}{b}\right)\ \sin\left(\arcsin\left(\frac{x}{b}\right)\right)}{2} + \frac{\arcsin\left(\frac{x}{b}\right)}{2}\right]$$

$$= b^2\left[\frac{\sqrt{1-\frac{x^2}{b^2}}\cdot\frac{x}{b}}{2} - \frac{\sin^{-1}\left(\frac{x}{b}\right)}{2}\right]$$

$$= \left.\frac{b^2\ \sin^{-1}\left(\frac{x}{b}\right)}{2} + \frac{bx\sqrt{1-\frac{x^2}{b^2}}}{2}\right|_0^b$$

$$= \frac{b^2\cdot \pi/2}{2} + \frac{b^2}{2} \qquad = 2\left[\frac{b^2\ \pi/2}{2} + 0\right]$$

$$= \frac{\pi}{2}\left[\frac{b^2}{2} + \frac{b^2}{2}\right] \qquad = b^2\ \pi/2$$

$$= 0.5\,\pi b^2$$

$$= 0.5\,\pi b^2$$

Question 8

Prove $\int h^{-1}(x)\,dx = xh^{-1}(x) - \int x(h^{-1}(x))'\,dx$

$$I = \int h^{-1}(x)\,dx$$

Applying integration by parts:

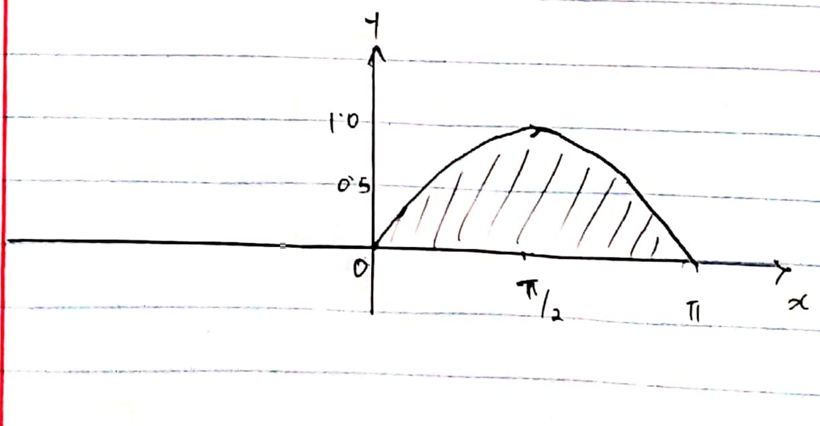
$$I = h^{-1}(x)\int dx - \int \frac{d}{dx} h^{-1}(x) \int dx$$

$$= h^{-1}(x)\cdot x - \int (h^{-1}(x))' \cdot x\,dx$$

Thus $\int h^{-1}(x)\,dx = x h^{-1}(x) - \int x(h^{-1}(x))'\,dx$

Question 9

$f(x) = \sin^2(x)$ and $g(x) = 0$ $x \in [0, \pi]$



$$\text{Area} = \int_0^{\pi} \sin^2(x)\,dx = 2\int_0^{\pi/2} \sin^2(x)\,dx$$

Take $\int \sin^2(x)\,dx$

Applying reduction formula: $\int \sin^n(x)\,dx = \frac{n-1}{n}\int \sin^{n-2}(x)\,dx - \frac{\cos(x)\sin^{n-1}(x)}{n}$

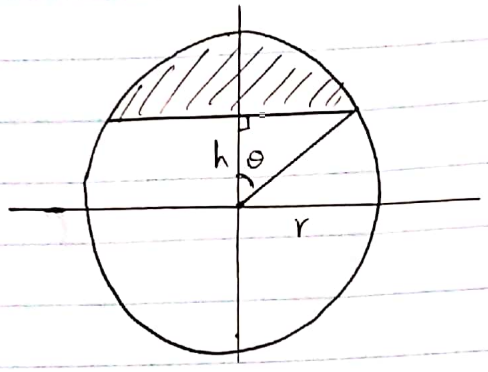
$$\int \sin^2(x)\,dx = -\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int \sin^0(x)\,dx$$

$$= 2\left[\frac{x}{2} - \frac{\cos(x)\sin(x)}{2}\right]_0^{\pi/2}$$

$$= 2\left[\frac{\pi}{4} - \frac{0}{2}\right]$$

$$= \frac{\pi}{2} \text{ square units}$$

Question 10



$x^2 + y^2 = r^2$

$y = \sqrt{r^2 - x^2}$

$x = \sqrt{r^2 - y^2}$

Area of semicircle $= \frac{\pi r^2}{2}$

Area of unshaded part $= 2\int_0^h \sqrt{r^2 - y^2}\, dy$

Take $\int \sqrt{r^2 - y^2}\, dy$

$y = r\sin(u)$; $u = \sin^{-1}(y/r)$, $dy = r\cos(u)\, du$

$= \int r\cos(u) \sqrt{r^2(1 - \sin^2(u))}\, du \quad = \int r^2 \cos u \sqrt{1 - \sin^2(u)}$

$= r^2 \int \cos^2(u)$

Applying reduction formula:

$\int \cos^2(u)\, du = \frac{\cos(u)\sin(u)}{2} + \frac{1}{2}\int du$

$= \frac{\cos(u)\sin(u)}{2} + \frac{u}{2}$

$= \frac{r^2 \cos(u)\sin(u)}{2} + \frac{r^2 u}{2}$

Substituting back $u = \sin^{-1}(y/r)$

$$\cos\left(\sin^{-1}\left(\frac{y}{r}\right)\right) = \sqrt{1-\frac{y^2}{r^2}}$$

$$\sin\left(\sin^{-1}\left(\frac{y}{r}\right)\right) = \frac{y}{r}$$

$$= \frac{ry\sqrt{1-y^2/r^2}}{2} + \frac{r^2\sin^{-1}(y/r)}{2}$$

$$\text{Area} = 2\left[\frac{ry\sqrt{1-y^2/r^2}}{2} - \frac{r^2\sin^{-1}(y/r)}{2}\right]_0^h$$

$$= 2\left[\frac{hr\cdot\sqrt{1-h^2/r^2}}{2} - \frac{r^2\sin^{-1}(h/r)}{2}\right]$$

Area of shaded part:

$$\frac{\pi r^2}{2} - 2\left[\frac{hr\cdot\sqrt{1-h^2/r^2}}{2} - \frac{r^2\sin^{-1}(h/r)}{2}\right]$$

$$= \frac{\pi r^2}{2} - \left(hr\cdot\sqrt{1-h^2/r^2} - r^2\sin^{-1}(h/r)\right)$$