

#35567 – linear algebra

la.

$$-2x_2 = 0 \quad x_2 = 0$$

$$\Rightarrow x_1 - x_2 - 3x_3 = 0$$

$$x_1 = 3x_3$$

$$\text{Let } x_3 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow A = P D P^{-1}$$

$$\text{or } D = P^{-1} A P$$

$$\text{where } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

la1.

Given that $p = -2 + x + 3x^2$ and $q = 4 - 7x^2$.

So that standard inner product of p and q is

$$\langle p, q \rangle = (-2) \times 4 + 1 \times 0 + 3 \times (-7) = -8 - 21 = -29$$

$$= -29$$

la2.

Let $\mathbb{R}^4$ have the Euclidean inner product and consider the vectors $u = (2, 1, -4, 0)$, $v = (-1, -1, 2, 2)$, and $w = (3, 2, 5, 4)$.

Let $x = (x_1, x_2, x_3, x_4)$ be a vector which is orthogonal to u, v and w .

Then we have that $\langle x, u \rangle = 2x_1 + x_2 - 4x_3 = 0$, $\langle x, v \rangle = -x_1 - x_2 + 2x_3 + 2x_4 = 0$ and $\langle x, w \rangle = 3x_1 + 2x_2 + 5x_3 + 4x_4 = 0$.

Since $2x_1 + x_2 - 4x_3 = 0$, $-x_1 - x_2 + 2x_3 + 2x_4 = 0$ we have that $2x_1 + x_2 - 4x_3 + 2(-x_1 - x_2 + 2x_3 + 2x_4) = 0 + 2 \cdot 0$ and $3(-x_1 - x_2 + 2x_3 + 2x_4) + 3x_1 + 2x_2 + 5x_3 + 4x_4 = 3 \cdot 0 + 0$ which gives us that $-x_2 + 4x_4 = 0$ and then $-x_2 + 11x_3 + 10x_4 = 0$.

Thus $x_2 = 4x_4$ and $x_3 = -\frac{6x_4}{11}$ and therefore since $2x_1 + x_2 - 4x_3 = 0$ we have that $x_1 = -\frac{34x_4}{11}$.

Hence $x = (-\frac{34x_4}{11}, 4x_4, -\frac{6x_4}{11}, x_4)$.

Now in order for x to be unitary we must have $\|x\| = 1$ and then we must have $\sqrt{(-\frac{34x_4}{11})^2 + (4x_4)^2 + (-\frac{6x_4}{11})^2 + x_4^2} = 1$.

Since $\sqrt{(-\frac{34x_4}{11})^2 + (4x_4)^2 + (-\frac{6x_4}{11})^2 + x_4^2} = \sqrt{\frac{1,156}{121}x_4^2 + 16x_4^2 + \frac{36}{121}x_4^2 + x_4^2} = \sqrt{(\frac{1,156}{121} + 16 + \frac{36}{121} + 1)x_4^2} = \sqrt{(\frac{3,249}{121})x_4^2} = \frac{57}{11}|x_4|$ we must have $\frac{57}{11}|x_4| = 1$ and which gives us $|x_4| = \frac{11}{57}$ and then $x_4 = -\frac{11}{57}$ or $x_4 = \frac{11}{57}$.

Thus $x = (\frac{34}{57}, -\frac{44}{57}, \frac{6}{57}, -\frac{11}{57})$ and $x' = (-\frac{34}{57}, \frac{44}{57}, -\frac{6}{57}, \frac{11}{57})$ are two unit vectors which are orthogonal to u, v and w .

la3.

$$\begin{aligned}
 2. \lambda_2 = 2 \\
 A - \lambda \times E &= \begin{pmatrix} 0 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \\
 A - \lambda \times E &= O, \text{ So we have a homogeneous system of linear equations, we solve it by Gaussian Elimination:} \\
 \begin{pmatrix} 0 & -1 & -1 & | & 0 \\ 1 & -2 & -1 & | & 0 \\ -1 & 1 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & -2 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ -1 & 1 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_3 - (-1) \times R_1 \rightarrow R_3} \begin{pmatrix} 1 & -2 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & -2 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 0 & -1 & -1 & | & 0 \end{pmatrix} \xrightarrow{R_2 / (-1) \rightarrow R_2} \begin{pmatrix} 1 & -2 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & -1 & -1 & | & 0 \end{pmatrix} \xrightarrow{R_3 + (-1) \times R_2 \rightarrow R_3} \begin{pmatrix} 1 & -2 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & -2 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1 + 2 \times R_2 \rightarrow R_1} \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{R_1 - R_2 \rightarrow R_1} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \\
 \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases} \quad (1) \\
 \begin{aligned}
 &\bullet \text{ Find the variable } x_2 \text{ from the equation 2 of the system (1):} \\
 &\quad x_2 = -x_3 \\
 &\bullet \text{ Find the variable } x_1 \text{ from the equation 1 of the system (1):} \\
 &\quad x_1 = -x_3 \\
 X &= \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} \\
 \text{Let } x_3 &= 1, v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}
 \end{aligned}
 \end{aligned}$$

la4.

Soln

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Find the eigen values

$$\det(A - \lambda I) = 0$$

$$\det \left(\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\det \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

The eigenvalues are

$$\lambda = 1, 2, 3$$

The diagonal composed of the Eigenvalues.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the Eigen Vector

For $\lambda = 1$

$$\begin{bmatrix} -2 & 4 & -2 \\ -3 & 3 & 0 \\ -3 & 1 & 2 \end{bmatrix}$$

Reduce the matrix row echelon form

$$(A - \lambda I) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This reduces the following system of equation

$$(A - \lambda I) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x - z &= 0 \\ y - z &= 0 \end{aligned}$$

Plug into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$v = \begin{bmatrix} z \\ z \\ z \end{bmatrix} \quad z \neq 0$$

Let $z = 1$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}$$

for $\lambda = 3$: $\begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$

The Eigen vectors are

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2/3 & 1/4 \\ 1 & 1 & 3/4 \\ 1 & 1 & 1 \end{bmatrix}$$

find P^{-1}

$$= \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix}$$

~~if~~ PDP^{-1}

$$P = \begin{bmatrix} 1 & 2/3 & 1/4 \\ 1 & 1 & 3/4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 3 & -5 & 3 \\ -3 & 9 & -6 \\ 0 & -4 & 4 \end{bmatrix}$$

la5.

Answered above in la4.

la6.

a. (p,q)

b. $d(p,q)$

c. $||p||$

d. $||q||$

$$a) (p, q) = \int_{-1}^1 p(x)q(x) dx$$

$$b) d(p, q) = \|p - q\|$$

$$c) \|p\| = \sqrt{\int_{-1}^1 p^2(x) dx}$$

$$d) \|p\| = \sqrt{\int_{-1}^1 q^2(x) dx}$$

la7.

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

la8.

(a)

$$T: V \rightarrow W$$

$$T(v) = \text{proj}_W(v)$$

By definition of projection onto W we have

$$\text{proj}_W v = \text{proj}_W (x+y) = x.$$

For two arbitrary scalars λ_i and two vectors v_i $v_i \in V$ we have

$$v_i = x_i + y_i \text{ with } x_i \in W \text{ and } y_i \in W^\perp$$

$\therefore$

$$\begin{aligned} \lambda_1 v_1 + \lambda_2 v_2 &= \lambda_1 (x_1 + y_1) + \lambda_2 (x_2 + y_2) \\ &= (\lambda_1 x_1 + \lambda_2 x_2) + (\lambda_1 y_1 + \lambda_2 y_2) \end{aligned}$$

Since W and $W^\perp$ are closed

$$\therefore \lambda_1 x_1 + \lambda_2 x_2 \in W$$

$$\text{and } \lambda_1 y_1 + \lambda_2 y_2 \in W^\perp$$

$$\Rightarrow \text{proj}_W (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 x_1 + \lambda_2 x_2$$

on the other hand

$$\text{we have } \lambda_i \text{proj}_W(v_i) = \lambda_i x_i$$

so

$$\begin{aligned} \text{proj}_W (\lambda_1 v_1 + \lambda_2 v_2) &= \lambda_1 x_1 + \lambda_2 x_2 \\ &= \lambda_1 \text{proj}_W(v_1) + \lambda_2 \text{proj}_W(v_2) \end{aligned}$$

This means proj_W is linear on V .

la9.

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the inner product of the vectors $\mathbf{u}$ and $\mathbf{v}$. This operator projects the vector $\mathbf{v}$ orthogonally onto the vector $\mathbf{u}$.

The Gram–Schmidt process then works as follows:

	$\mathbf{u}_1 = \mathbf{v}_1,$		$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\ \mathbf{u}_1\ }$
	$\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2),$		$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\ \mathbf{u}_2\ }$
	$\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3),$		$\mathbf{e}_3 = \frac{\mathbf{u}_3}{\ \mathbf{u}_3\ }$

keeping these things in view,

$$\mathbf{v}_1 = \mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{(1, 1, 1)}{\sqrt{1+1+1}} = \frac{(1, 1, 1)}{\sqrt{3}}$$

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u}_2 - (\mathbf{u}_2, \mathbf{w}_1) \mathbf{w}_1 = \begin{pmatrix} -1, 1, 0 \end{pmatrix} - \left(\begin{pmatrix} -1, 1, 0 \end{pmatrix}, \frac{(1, 1, 1)}{\sqrt{3}} \right) \left(\frac{(1, 1, 1)}{\sqrt{3}} \right) \\ &= (-1, 1, 0) \end{aligned}$$

$$\text{so, } \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{(-1, 1, 0)}{\sqrt{2}}$$

$$\begin{aligned}
 v_3 &= \left(1, 2, 1\right) - \left(\left(1, 2, 1\right), \frac{(1, 1, 1)}{\sqrt{3}}\right) \frac{(1, 1, 1)}{\sqrt{3}} - \left(\left(1, 2, 1\right), \frac{(-1, 1, 0)}{\sqrt{2}}\right) \frac{(-1, 1, 0)}{\sqrt{2}} \\
 &= \left(1, 2, 1\right) - \frac{(4, 4, 4)}{3} + \frac{(-1, 1, 0)}{2} = \frac{(6, 12, 6) - (8, 8, 8) + (-3, 3, 0)}{6} \\
 &= \frac{(-5, 7, -2)}{6}
 \end{aligned}$$

so,

now, the required orthonormal basis for the given basis is $\{w_1, w_2, w_3\}$

la10.

Ans. Given matrix

$$\text{Say } A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Char eq. of matrix A is $|A - \lambda I| = 0$

which is obtained by.

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 12\lambda - 16 = 0$$

Hence eigen values corresponding to the given matrix are solution of char. eq.

which are $\lambda = -2, -2, 4$

Now ~~eigen vector~~ to find the basis of
eigen space corresponding to matrix

for eigen value $\lambda = -2$

$$[A - \lambda I][X] = 0$$

$$\text{We have } \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we have

$$x - y + z = 0$$

$$\text{for } y=1, z=0 \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{for } y=0, z=1$$

$$X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Here X_1 and X_2 are the basis correspond
to $\lambda = -2$

Now for $\lambda = 4$

$$[A - \lambda I]X = 0$$

We have
$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system

$$-x - y + z = 0$$

$$-6y + z = 0$$

For $z = 1$ $X_3 = \begin{bmatrix} -5/6 \\ 1/6 \\ 1 \end{bmatrix}$

Hence X_1, X_2, X_3 are basis corresponding to A .
