

The Man Who Knew Infinity

[Enter your name here]

MATH-199 Math/Stat Seminar

Assignment #5 (The second L^AT_EX assignment)

[Enter the date here]

Srinivasa Ramanujan (1887-1920) was an Indian mathematician that had almost no formal training in mathematics, but made notable contributions to the fields of continued fractions, infinite series, number theory, and mathematical analysis. Initially working in isolation in India, he quickly became recognized by fellow Indian mathematicians. After becoming known to the wider mathematical community, Ramanujan was mentored by English mathematician G. H. Hardy (1877-1947), a professor at Cambridge, first through correspondence and second through visits to England to converse with Hardy and other top mathematicians of the time. Throughout his lifetime, Ramanujan produced nearly 4,000 results (some of which were previously unknown). Nearly all of the results have been verified to be correct. Today, most mathematicians consider Ramanujan one of the greatest Indian mathematicians in history.

1 Early Life

Srinivasa Ramanujan (see Figure 1) was born on 22 December 1887 in Erode, located in the southern Indian state of Tamil Nadu.

He began to focus on mathematics at an early age, and, at the age of about fifteen, borrowed a copy of G.S. Carr's *Synopsis of Pure and Applied Mathematics* [3], which served as his primary source for learning mathematics. At about the time Ramanujan entered college, he began to record his mathematical discoveries in notebooks.

Living in poverty with no means of financial support, suffering at times from serious illnesses, and working in isolation, Ramanujan devoted his efforts to mathematics and continued to record his discoveries without proofs in notebooks for the next six years.



Figure 1: Photo of Ramanujan taken in his late twenties.

2 Elementary Mathematics

Many of Ramanujan's discoveries can be appreciated by those with only a knowledge of high school algebra. Chapter 2 in the second notebook, the unorganized portions of the second and third notebooks, and the problems that Ramanujan submitted to the *Journal of the Indian Mathematical Society* [5] are excellent sources for these gems.

Ramanujan enjoyed finding equal sums of powers. For example, in [2], he shows if $a + b + c = 0$, then

$$2(ab + ac + bc)^4 = a^4(b - c)^4 + b^4(a - c)^4 + c^4(a - b)^4. \quad (1)$$

In fact, in his third notebook [5, p. 96], Ramanujan recorded similar formulas for $2(ab + ac + bc)^{2n}$, when $n \in \{1, 2, 3, 4\}$, and wrote “and so on” to indicate that he possessed a general procedure for finding such formulas.

Ramanujan was fond of stating intriguing formulas such as

$$2 \sin \left(\frac{\pi}{18} \right) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \dots}}}} \quad (2)$$

or

$$\sqrt[3]{\cos 40^\circ} + \sqrt[3]{\cos 80^\circ} - \sqrt[3]{\cos 20^\circ} = \sqrt[3]{\frac{3}{2} \left(\sqrt[3]{9} - 2 \right)}. \quad (3)$$

In most instances, these are special cases of more general theorems that he established.

3 His Notebooks

To provide a feeling for Ramanujan’s notebooks, we reproduce here (see Figure 2) a page from his third notebook, Chapter XVIII [1]. There is no particular reason for choosing this page except that it contains an entry which has become a ‘folklore’ which illustrates the attachment Ramanujan had to numbers prompting British mathematician John E. Littlewood (1885-1977) to state that to Ramanujan every number is a personal friend.

There are two parts shown in this provided page. The first part is concerning the geometrical construction of a square whose area is equal to that of a given circle. Here he gives a geometrical construction for finding the length of the side of a square whose area equals that of the circle. He also reproduced this and another geometrical construction for π in his later paper on “Modular equations and approximations to π ”. In this paper he deduced several approximations and formulae for π such as

$$\pi \approx \frac{63}{25} \left(\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}} \right), \quad (4)$$

$$\pi \approx 12\sqrt{190} \log \left\{ \left(2\sqrt{2} + \sqrt{10} \right) \left(3 + \sqrt{10} \right) \right\}, \quad (5)$$

and

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}. \quad (6)$$

The expression in (4) gives the value of π accurate to 9 decimal places and the one in (5) has an accuracy of 18 decimal places. Ramanujan asserted that (6) is the most intriguing one because it is a very “rapidly convergent” series. In 1986, two computer scientists used a version of Ramanujan’s formula to calculate π to 17 million digits and found that the formula converges to the exact value with far greater efficiency than any previous method. This success proved that Ramanujan’s insight was correct.

4 The Hardy-Ramanujan Number

One of the most famous stories related to Ramanujan is the *Hardy-Ramanujan number*. Hardy was visiting Ramanujan in a hospital, and, as told by Hardy himself:

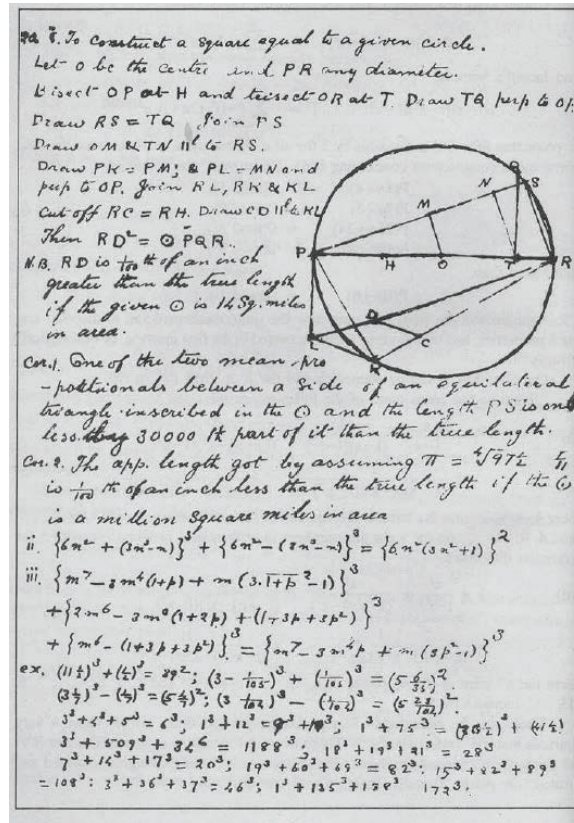


Figure 2: Ramanujan's handwritten page from his *Notebook*.

“I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. “No,” he replied, “it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.”

That is, $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

This anecdote reflects the extent of how well Ramanujan knew and studied the integers to have been aware and able to repeat this property while in the hospital.

Generalizations of these numbers are now called *taxicab numbers*.

5 Prime Numbers

A sample of a major result by Ramanujan is in his study of the prime numbers and is summarized in [6].

One of the most studied functions in number theory is the *prime counting* function. For a real number x , we denote by $\pi(x)$ the number of prime numbers (i.e., those positive integers divisible by exactly two integers) less than or equal to x . So, for example, $\pi(10) = 4$ because 2, 3, 5, 7 are all the primes less than 10; it is left to the reader to verify that $\pi(100) = 25$.

Definition 1. For $n \geq 1$, the n -th *Ramanujan prime* is the smallest positive integer R_n with the property that if $x \geq n$, then $\pi(x) - \pi(\frac{1}{2}x) \geq n$.

The sequence of Ramanujan primes is 2, 11, 17, 29, 41, $\dots$. A bound on the Ramanujan primes is given in the following theorem.

Theorem 2. *The n -th Ramanujan prime satisfies the inequalities*

$$2n \log 2n < R_n < 4n \log 4n \quad (n \geq 1). \quad (7)$$

Furthermore, if p_n denotes the n -th prime, then $p_{2n} < R_n < p_{4n}$, for $n > 1$.

6 Partition Functions

Another area of number theory in which Ramanujan had worked extensively and is well-known for his contributions is the theory of *partition functions*. The partition function $p(n)$ gives the number of ways of writing an integer n as a sum of positive integers. Thus 4 can be written in five different ways (without regard to order), as

$$4, 3 + 1, 2 + 2, 2 + 1 + 1, \text{ and } 1 + 1 + 1 + 1. \quad (8)$$

The partition number $p(4)$ is therefore 5. The partition numbers of the integers 1 through 10 are given in Table 1.

Number	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42

Table 1: The number of partitions for the integers 1 through 10.

There is no formula that provides the values of $p(n)$ for a given n , but the numbers can be determined through *generating functions*.

7 Awards and Recognitions

Ramanujan has been recognized after his death with the following honors:

- Ramanujan's home state (Tamil Nadu) celebrates Ramanujan's birthday (December 22) as a special holiday.
- The government of India in 1962 commemorated his life and achievements in number theory by issuing a stamp with his likeness.
- A prize for young mathematicians was created in Ramanujan's name by the International Centre for Theoretical Physics and the International Mathematical Union.
- SASTRA University has established a prize of \$10,000 to be awarded annually to a mathematician less than 32 years old for outstanding contributions in an area of mathematics influenced by Ramanujan.

References

- [1] Bruce C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.
- [2] Bruce C. Berndt, *Ramanujan's Notebooks, Part IV*, Springer-Verlag, New York, 1994.
- [3] George S. Carr, *A Synopsis of Elementary Results in Pure and Applied Mathematics* (2 Volumes), Cambridge University Press, 2013.
- [4] Robert Kanigel, *The Man Who Knew Infinity: A Life of the Genius Ramanujan*, Fifth edition, Washington Square Press, 1992.
- [5] Srinivasa Ramanujan, *Collected Papers*, American Mathematical Society, Providence, 2000.
- [6] Jonathan Sondow, Ramanujan Primes and Bertrand's Postulate, *The American Mathematical Monthly*, Vol. 116, No. 7 (Aug. - Sep., 2009), pp. 630-635.