

1.

(a)

$$\frac{dS}{dt} = -\beta SI \leq 0 \text{ for } \beta > 0, S \geq 0, I \geq 0$$

$S(t)$ is decreasing, i.e. monotone on $[0, \infty)$.

$S(t) \leq N$, so $S(t)$ is bounded.

So S_∞ exists.

$$\frac{dR}{dt} = \nu I \geq 0 \text{ for } \nu > 0, I \geq 0$$

$R(t)$ is increasing, i.e. monotone on $[0, \infty)$.

$R(t) \leq N$, so $R(t)$ is bounded.

So R_∞ exists.

$$S(t) + I(t) + R(t) = N$$

$$\lim_{t \rightarrow \infty} (S(t) + I(t) + R(t)) = \lim_{t \rightarrow \infty} N$$

$$\lim_{t \rightarrow \infty} S(t) + \lim_{t \rightarrow \infty} I(t) + \lim_{t \rightarrow \infty} R(t) = N$$

$$S_\infty + I_\infty + R_\infty = N$$

$$I_\infty = N - S_\infty - R_\infty$$

Since S_∞ and R_∞ exist, so $N - S_\infty - R_\infty$ exists, hence **I_∞ exists.**

(b)

Since $\lim_{t \rightarrow \infty} R(t) = R_\infty \neq \infty$, $\lim_{t \rightarrow \infty} R'(t) = 0$.

$$\lim_{t \rightarrow \infty} R'(t) = \lim_{t \rightarrow \infty} (\nu I) = 0$$

$$\nu \lim_{t \rightarrow \infty} I = 0$$

$\nu I_\infty = 0$, and $\nu > 0$, so

$$I_\infty = \mathbf{0}$$

2.

(a)

Since S is a decreasing function, $S(t + 1) < S(t)$ for all t :

So, $S(t) \leq S(0)$ for all $t \rightarrow \beta S(t) - \nu \leq \beta S(0) - \nu < 0$ for all t .

Hence, if $\beta S(0) - \nu < 0$, then $\beta S(t) - \nu < 0$.

$$\frac{dI}{dt} = \beta SI - \nu I = I(\beta S - \nu)$$

For $I \geq 0$ and $\beta S - \nu < 0$: $I(\beta S - \nu) < 0$,

Hence:

$$\frac{dI}{dt} < 0$$

(b)

Let $\beta S(0) - v = a$, where a is some constant such that $a > 0$, i.e. $\beta S(0) - v > 0$.

Since S is a decreasing function, $S(t + 1) < S(t)$ for all t :

Then, $S(t) \leq S(0)$ for all $t \rightarrow \beta S(t) - v \leq a, a > 0$ for all t .

Let t_0 be a time such that $\beta S(t) - v = 0 < a$.

Then, it must be that $t_0 > 0$, because $\beta S(t) - v < a = \beta S(0) - v$, i.e.:

$\beta S(t) - v < \beta S(0) - v$, which is only true for some $t > 0$.

Now at $t = t_0$, $\beta S(t) - v = 0$, so:

$$\frac{dI}{dt}|_{t=t_0} = I(\beta S(t_0) - v) = I(0) = 0$$

Hence, $I(t)$ has a stationary point at t_0 .

For $0 < t < t_0$, $0 < \beta S(t) - v < a$, because $S(t)$ is a decreasing function, so $\beta S(t) - v > 0$ and $I \geq 0$, so $I(\beta S(t_0) - v) \geq 0$ for $t < t_0$, or simple, I is increasing at all times before t_0 .

Hence, the stationary point at t_0 is a maximum point, and **I rises to a maximum value at a $t > t_0$, after which it must decrease.**

Since for $t > t_0$, $\beta S(t) - v < 0$, and $I > 0$ for $t < t_e$ for some $t_e > t_0$ at which I is 0, then:

$$\frac{dI}{dt} < 0 \text{ for all } t_0 < t < t_e$$

where

Hence, after t_0 , I will only decrease (until it reaches zero at t_e).

3.

(a)

$$\frac{dI}{dS} = \frac{\frac{dI}{dt}}{\frac{dS}{dt}} = \frac{\beta SI - \nu I}{-\beta SI}$$

$$\frac{dI}{dS} = \frac{I(\beta S - \nu)}{-\beta SI} = \frac{\nu - \beta S}{\beta S} = \frac{\nu}{\beta S} - 1$$

$$\int_{I(0)}^{I(t)} dI = \int_{S(0)}^{S(t)} \frac{\nu}{\beta S} - 1 \, dS$$

$$[I]_{I(0)}^{I(t)} = \left[\frac{\nu}{\beta} \ln S - S \right]_{S(0)}^{S(t)}$$

$$I(t) - I(0) = \left(\frac{\nu}{\beta} \ln S(t) - S(t) \right) - \left(\frac{\nu}{\beta} \ln S(0) - S(0) \right)$$

$$I(t) - I(0) = \left(\frac{\nu}{\beta} \ln S(t) - S(t) \right) - \left(\frac{\nu}{\beta} \ln S(0) - S(0) \right)$$

$$I(t) = -S(t) + \frac{\nu}{\beta} \ln S(t) + I(0) + S(0) - \frac{\nu}{\beta} \ln S(0)$$

(b)

$$\frac{dI}{dt} = \beta S(t_0)I - \nu I = 0$$

$$I(\beta S(t_0) - \nu) = 0$$

$$I = I_{max} \neq 0, \text{ so}$$

$$\beta S(t_0) - \nu = 0$$

$$S(t_0) = \frac{\nu}{\beta}$$

$$I(t_0) = -S(t_0) + \frac{\nu}{\beta} \ln S(t_0) + I(0) + S(0) - \frac{\nu}{\beta} \ln S(0)$$

$$I_{max} = -\frac{\nu}{\beta} + \frac{\nu}{\beta} \ln \frac{\nu}{\beta} + I(0) + S(0) - \frac{\nu}{\beta} \ln S(0)$$

$$I_{max} = I(0) + S(0) - \frac{\nu}{\beta} + \left(\frac{\nu}{\beta} \ln \frac{\nu}{\beta} \right) - \left(\frac{\nu}{\beta} \ln S(0) \right)$$