chapter 3 : Polynomial approximation and interpolation

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Problem: Given a function $f(x)$, find a polynomial approximation $p_n(x)$

Approximation: $\int_a^b f(x)dx \to \int_a^b p_n(x)dx$

Why polynomial?

1. Weierstrass Approximation Thorem

Suppose that f is defined and continuous on $[a,b]$. For each $\epsilon > 0$, there exists a polynomial $p_n(x)$ defined on $[a,b]$, with property that $|f(x) - p_n(x)| < \epsilon$ for all $x \in [a,b]$

2. simple $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$: standard form

One solution: The Taylor polynomial of degree n about a point $x = c$

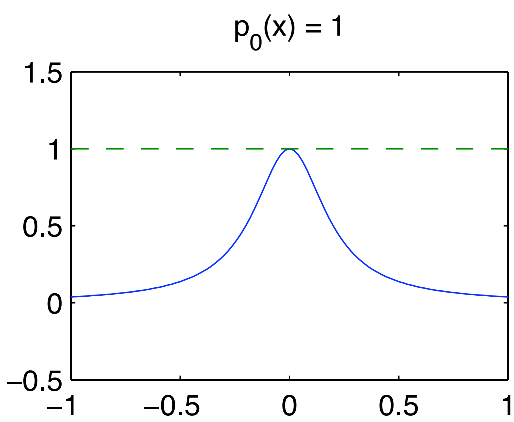
$$p_n(x) \ = \ f(c) + f'(c)(x-c) + \tfrac{1}{2}f''(c)(x-c)^2 + \cdots + \tfrac{1}{n!}f^{(n)}(c)(x-c)^n.$$

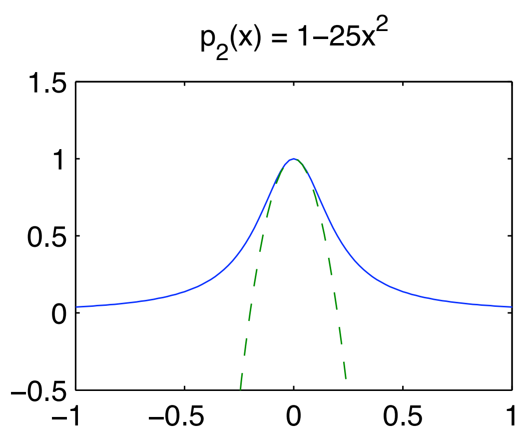
ex : $f(x) = \dfrac{1}{1+25x^2}$, $c = 0$, $p_n(x) = ?$

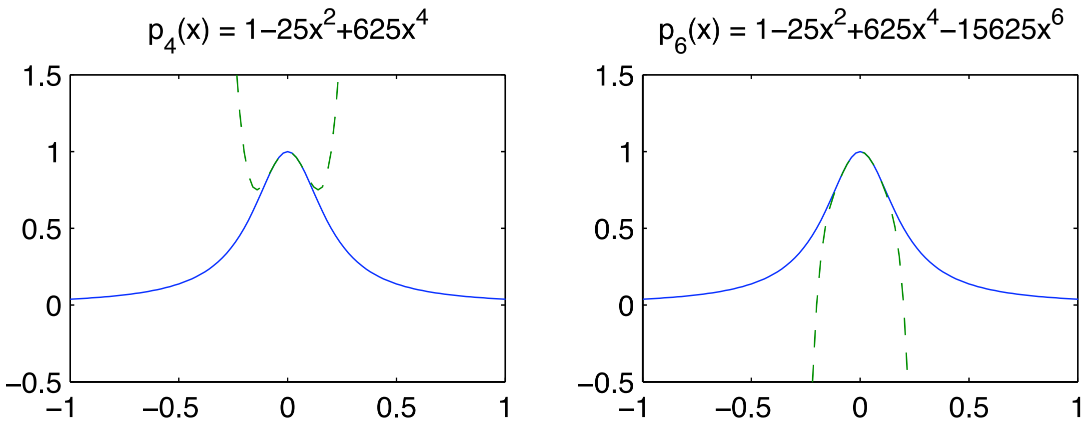
In this case we can find $p_n(x)$ without computing $f(c), f'(c), \ldots, f^{(n)}(c)$.

recall the geometric series : $\dfrac{1}{1-r} = 1 + r + r^2 + \cdots$, converges for $|r| < 1$

$$\frac{1}{1+25x^2} = \frac{1}{1-(-25x^2)} = 1 + (-25x^2) + (-25x^2)^2 + \cdots \text{ , converges for } |x| \le \tfrac{1}{5}$$







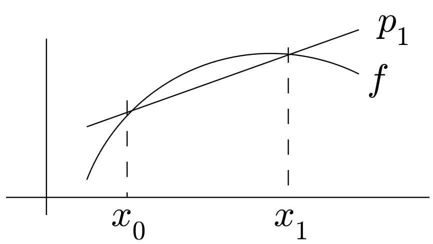
The Taylor polynomial $p_n(x)$ is a good approximation to $f(x)$ when x is close to c, but in general we need to consider other methods of approximation.

3.2 Lagrange Polynomials

thm: Assume $f(x)$ is given and let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points. Then there exists a unique polynomial $p_n(x)$ of degree $\le n$ which interpolates $f(x)$ at the given points, i.e. such that $p_n(x_i) = f(x_i)$ for $i = 0 : n$.

pf : omit

ex : $n = 1 \Rightarrow x_0\,,\, x_1$



$$p_1(x) = f(x_0) + \left(\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right)(x - x_0)$$

$$\text{check : } \begin{cases} \deg p_1 \le 1\,, \\ p_1(x_0) = f(x_0)\,,\ p_1(x_1) = f(x_1) \quad \underline{\text{ok}} \end{cases}$$

questions

1. What is the form of $p_n(x)$ for $n \ge 2$?

2. What is the best choice of the interpolation points $x_0, \ldots, x_n$?

Consider the construction of a polynomial of degree at most n that passes through the $n+1$ points

$$(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$$

Construct: for each $k = 0, 1, 2, ..., n$, a polynomial of degree n, $L_{n,k}(x)$, with property

$$L_{n,k}(x_i) = 0 \text{ when } i \ne k, = 1 \text{ when } i = k$$

To satisfy $L_{n,k}(x_i) = 0$ for each $i = k$, $L_{n,k}(x)$ must contain the term

$$(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)$$

To satisfy $L_{n,k}(x_k) = 1$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

The polynomial that we are looking for is

$$p_n(x) = \Sigma_{k=0}^{n} f(x_k) L_{n,k}(x)$$

Let's check

1. it is nth order polynomial since $L_{n,k}$ is a nth order polynomial, $f(x_k)$ are constants

2. whether $p_n(x_k) = f(x_k)$?

$p_n(x) = f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + \cdots + f(x_n)L_{n,n}(x)$

$\Rightarrow p_n(x_k) = f(x_0)L_{n,0}(x_k)+f(x_1)L_{n,1}(x_k)+\cdots+f(x_{k-1})L_{n,k-1}(x_k)+f(x_k)L_{n,k}(x_k)+ f(x_{k+1})L_{n,k+1}(x_k) + \cdots + f(x_n)L_{n,n}(x_k)$

$= f(x_k)$

Summary

nth Lagrange Integrating Polynomial

$p_n(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$

where

$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$

notes

if (1) $x_0, x_1, \cdots, x_n,\ \ (n+1)$ distinct numbers

(2)f is a function whose values are given at these points: $f(x_i), i = 0, ..., n$ known

(3) $f(x_i) = P_n(x_i),\ \ i = 0, ..., n$

$\Rightarrow p_n(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$ is unique polynomial of degree at most n

e.g. a) Use the numbers (called nodes) $x_0 = 2, x_1 = 2.75$ and $x_2 = 4$ to find the second order Lagrange interpolating polynomial for $f(x) = 1/x$; b) Use first determine the coefficient polynomials $L_0(x), L_1(x)$ and $L_2(x)$

$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x - 2.75)(x - 4)$

$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x - 2)(x - 4)$

$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x - 2)(x - 2.75)$

also

$f(x_0) = f(2) = 1/2,\ \ f(x_1) = f(2.75) = 4/11,\ \ f(x_2) = f(4) = 1/4$

$p(x) = \sum_{k=0}^{2} f(x_k)L_k(x)$

$= \frac{1}{3}(x - 2.75)(x - 4) - \frac{64}{165}(x - 2)(x - 4) + \frac{1}{10}(x - 2)(x - 2.75)$

$= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$

b) An approximation $f(3) = 1/3$

$f(3) \approx p(3) = \frac{9}{22} - \frac{105}{88} + 49/44 = 29/88 \approx 0.32955$

eg. The outdoor temperature $T(t)$ is recorded at two-hour intervals starting at 8am and ending at 4pm, but the 2pm measurement was accidentally omitted. The recorded temperatures are $T(8am) = 30F, T(10am) = 40F, T(12pm) = 50F, T(4pm) = 60F$.

a) Construct the interpolating polynomial of the temperature using the given data.

b) Use the polynomial in part (a) to estimate the missing 2pm temperature.

Let $t = 0$ is $8am$, i.e. $T(0) = 30$, similarily

$t = 2$ is $10am$, i.e. $T(2) = 40$; $t = 4$ is $12am$, i.e. $T(4) = 50$

$t = 6$ is $2pm$, i.e. $T(6) =?$; $t = 8$ is $4pm$, i.e. $T(8) = 60$

we have $(0, T(0)), (2, T(2)), (4, T(4)), (8, T(8))$, it can determine a 3rd order polynomial

$L_0(t) = \frac{(t-2)(t-4)(t-8)}{(0-2)(0-4)(0-8)}$

$L_1(t) = \frac{(t-0)(t-4)(t-8)}{(2-0)(2-4)(2-8)}$

$L_2(t) = \frac{(t-0)(t-2)(t-8)}{(4-0)(4-2)(4-8)}$

$L_3(t) = \frac{(t-0)(t-2)(t-4)}{(8-0)(8-2)(8-4)}$

$p(t) = \sum_{k=0}^{3} T(t_k)L_k(t) = 30 * L_0(t) + 40 * L_1(t) + 50 * L_2(t) + 60 * L_3(t)$

$= -30 * \frac{(t-2)(t-4)(t-8)}{64} + 40 * \frac{(t)(t-4)(t-8)}{24} - 50 * \frac{(t)(t-2)(t-8)}{32} + 60 * \frac{(t)(t-2)(t-4)}{192}$

$\Rightarrow p(t = 6) = -30 * \frac{(6-2)(6-4)(6-8)}{64} + 40 * \frac{(6)(6-4)(6-8)}{24} - 50 * \frac{(6)(6-2)(6-8)}{32} + 60 * \frac{(6)(6-2)(6-4)}{192} = 57.5F$

Recall nth Taylor polynomial about $x = c$ has an error term of the form

$\frac{f^{n+1}(\xi)}{(n+1)!}(x-c)^{(n+1)}$, where ξ between x and c

<u>Lagrange Polynomial Error Formula</u>

$f(x) = p_n(x) + \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$

for some unknown number ξ lies in the smallest interval that contains $x_0, x_1, \cdots, x_n$ and x

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3.3 Divided Differences

note : The interpolating polynomial $p_n(x)$ can be written in different forms.

standard form

$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

Newton's form

$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$

note

1. The example above with $n = 1$ used Newton's form for $p_1(x)$.
2. The coefficients in each form are different; how can they be computed?

thm: The coefficients in Newton's form of $p_n(x)$ can be computed as follows.

$a_0 = f(x_0) = f[x_0]$

$a_1 = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1]$: 1st divided difference

$a_2 = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$: 2nd divided difference

...

$a_n = \dfrac{f[x_1, \dots, x_n] - f[x_0, \dots, x_{n-1}]}{x_n - x_0} = f[x_0, \dots, x_n]$: nth divided difference

3. It can be shown that nth Lagrange interpolation polynomial for f w.r.t. $x_0, x_1, \cdots, x_n$ can be expressed as

$p_n(x) = f[x_0] + f(x_0, x_1)(x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$: called Newton's divided-difference formula

pf. skip, just check $n = 0, n = 1$, and $n = 2$

$n = 0$, ok, because $p_0(x) = a_0, p_0(x_0) = f(x_0) \Rightarrow a_0 = f(x_0)$

$n = 1$, ok, because $p_1(x) = a_0 + a_1(x - x_0)$, we showed $a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ slope

$n = 2$: need to work

$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

need to show $a_0 = f[x_0], a_1 = f[x_0, x_1], a_2 = f[x_0, x_1, x_2]$

define $g(x) = \left(\dfrac{x - x_0}{x_2 - x_0}\right) q_1(x) + \left(\dfrac{x_2 - x}{x_2 - x_0}\right) p_1(x)$

where

$$p_1(x) = f[x_0] + f[x_0, x_1](x - x_0) \ : \ \text{interpolates } f(x) \text{ at } x_0\,,\, x_1$$

$$q_1(x) = f[x_1] + f[x_1, x_2](x - x_1) \ : \ \text{interpolates } f(x) \text{ at } x_1\,,\, x_2$$

then $g(x)$ has the following properties

1. $\deg g \leq 2$
2. $g(x_0) = \left(\frac{x_0-x_0}{x_2-x_0}\right) q_1(x_0) + \left(\frac{x_2-x_0}{x_2-x_0}\right) p_1(x_0) = p_1(x_0) = f(x_0)$

$$g(x_1) = \left(\frac{x_1 - x_0}{x_2 - x_0}\right) q_1(x_1) + \left(\frac{x_2 - x_1}{x_2 - x_0}\right) p_1(x_1)$$

$$= \left(\frac{x_1 - x_0}{x_2 - x_0}\right) \left(f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_1 - x_1)\right)$$

$$+ \left(\frac{x_2 - x_1}{x_2 - x_0}\right) \left(f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0)\right)$$

$$= \left(\frac{x_1 - x_0}{x_2 - x_0}\right) f(x_1) + \left(\frac{x_2 - x_1}{x_2 - x_0}\right) f(x_1)$$

$$= \frac{x_2 - x_0}{x_2 - x_0} f(x_1)$$

$$= f(x_1)$$

$$g(x_2) = \left(\frac{x_2 - x_0}{x_2 - x_0}\right) q_1(x_2) + \left(\frac{x_2 - x_2}{x_2 - x_0}\right) p_1(x)$$

$$= q_1(x_2)$$

$$= f(x_1) + \frac{f(x_1) - f(x_1)}{x_2 - x_1}(x_2 - x_1)$$

$$= f(x_2)$$

then $g(x) = p_2(x)$ for all x (by uniqueness theorem on polynomial interpolation)

note : the coefficient of x^2 in $g(x)$ is $\dfrac{f[x_1, x_2]}{x_2 - x_0} - \dfrac{f[x_0, x_1]}{x_2 - x_0}$

the coefficient of x^2 in $p_2(x)$ is a_2

$\Rightarrow a_2 = f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2]}{x_2 - x_0} - \dfrac{f[x_0, x_1]}{x_2 - x_0}$ as required

$n \geq 3$: follows the same way <u>ok</u>

<u>ex</u> : $n = 2 \Rightarrow x_0, x_1, x_2$

$$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$
$$= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

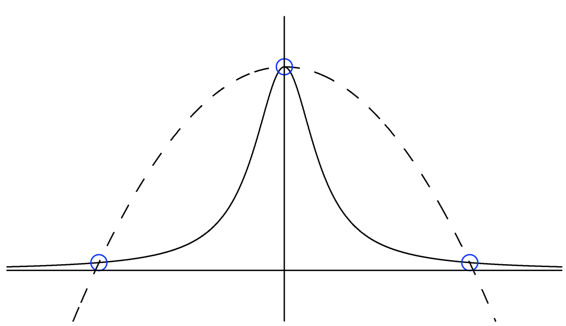
<u>divided difference table</u>

$$\begin{array}{llll}
x_0 \text{ --- } & f[x_0]^* & & \\
 & & f[x_0,x_1]^* & \\
x_1 \text{ --- } & f[x_1] & & f[x_0,x_1,x_2]^* \\
 & & f[x_1,x_2] & \\
x_2 \text{ --- } & f[x_2] & &
\end{array}$$

The starred values are the coefficients in Newton's form of $p_2(x)$.

<u>ex</u>

$$f(x) = \frac{1}{1 + 25x^2} \ , \ x_0 = -1 \ , \ x_1 = 0 \ , \ x_2 = 1 \Rightarrow p_2(x) = ?$$



$$\begin{array}{cccc}
x_i & f(x_i) & & \\
-1 \text{ --- } & \frac{1}{26} & & \\
 & & \frac{25}{26} & \\
0 \text{ --- } & 1 & & -\frac{25}{26} \\
 & & -\frac{25}{26} & \\
1 \text{ --- } & \frac{1}{26} & &
\end{array}$$

$$p_2(x) = \tfrac{1}{26} + \tfrac{25}{26}(x - (-1)) - \tfrac{25}{26}(x - (-1))(x - 0) \ : \ \text{Newton's form}$$
$$= \tfrac{1}{26} + \tfrac{25}{26}(x + 1) - \tfrac{25}{26}(x + 1)x$$
$$= 1 - \tfrac{25}{26}x^2 \ : \ \text{standard form}$$

check : $p_2(-1) = \frac{1}{26}$, $p_2(0) = 1$, $p_2(1) = \frac{1}{26}$ <u>ok</u>

$$\int_{-1}^{1} f(x)dx = 2\int_0^1 \frac{dx}{1 + 25x^2} = \cdots = 2 \cdot \tfrac{1}{5} \tan^{-1} 5x\Big|_0^1 = \tfrac{2}{5} \tan^{-1} 5 = 0.5494$$

$$\int_{-1}^{1} p_2(x)dx = 2\int_0^1 (1 - \tfrac{25}{26}x^2)dx = 2(x - \tfrac{25}{26} \cdot \tfrac{1}{3}x^3)\Big|_0^1 = 2(1 - \tfrac{25}{78}) = \tfrac{106}{78} = 1.3590$$

Hence $p_2(x)$ is a poor approximation to $f(x)$. Can we do better?

3.3.1 optimal interpolation points, not in the book

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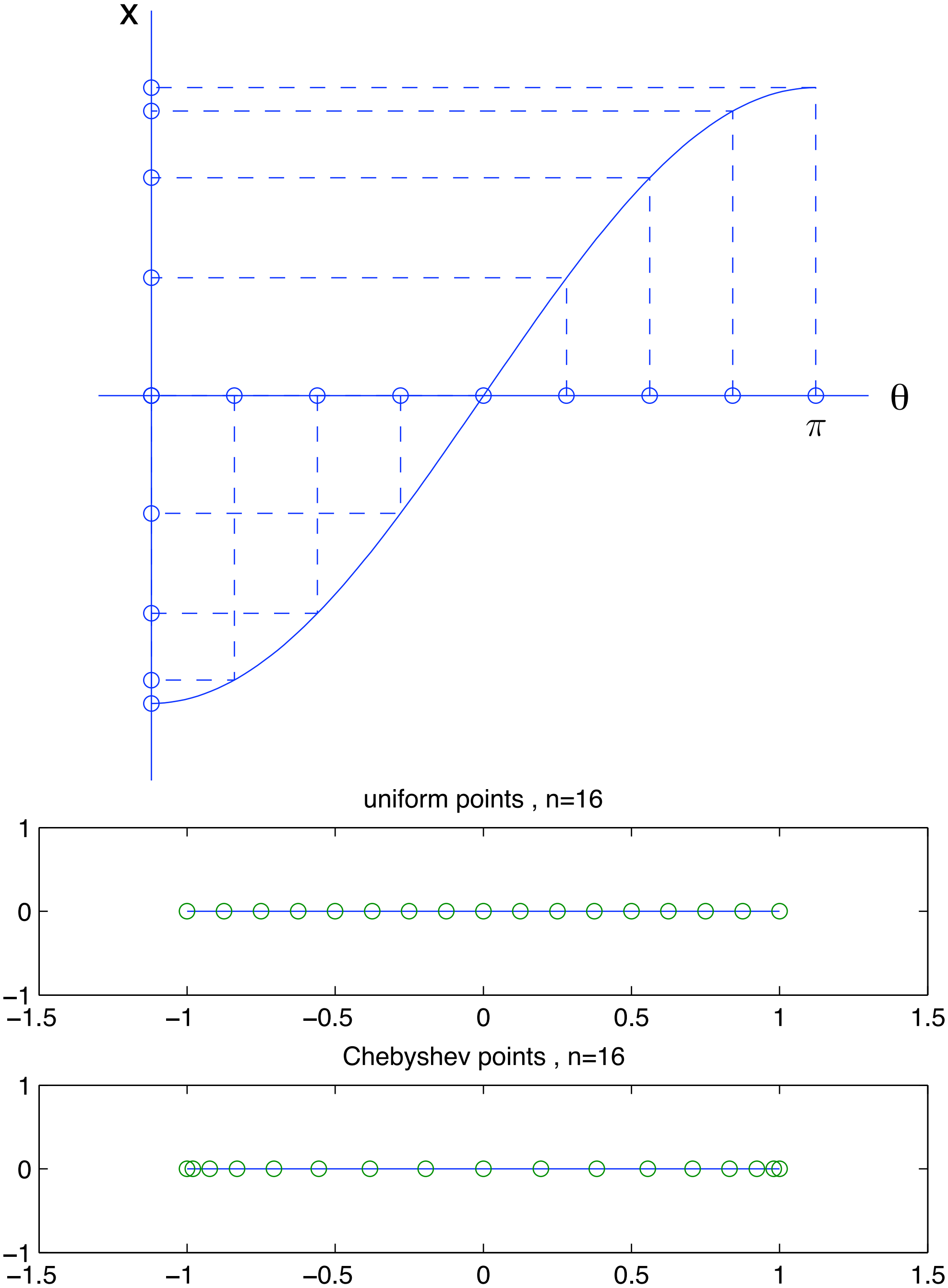
Given $f(x)$ for $-1 \leq x \leq 1$, how should the interpolation points $x_0, \ldots, x_n$ be chosen? Consider two options.

1. uniform points : $x_i = -1 + ih$, $h = \dfrac{2}{n}$, $i = 0 : n$

2. Chebyshev points : $x_i = -\cos\theta_i$, $\theta_i = ih$, $h = \dfrac{\pi}{n}$, $i = 0 : n$

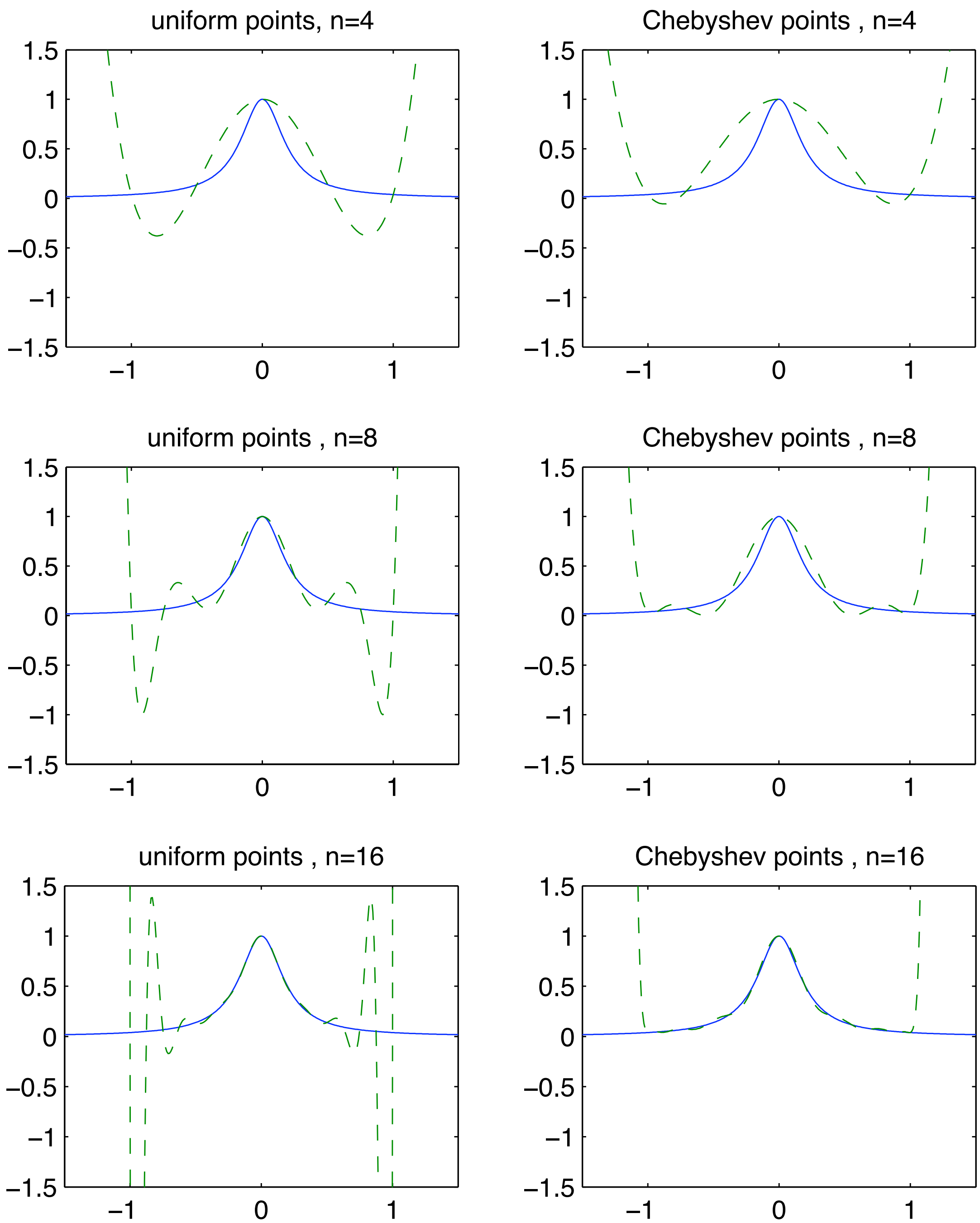


note : The Chebyshev points are clustered near the endpoints of the interval.

$\underline{\text{ex}}$: $f(x) = \dfrac{1}{1+25x^2}$, $-1 \le x \le 1$

solid line : $f(x)$, given function

dashed line : $p_n(x)$, interpolating polynomial



1. Interpolation at the uniform points gives a good approximation near the center of the interval, but it gives a bad approximation near the endpoints.

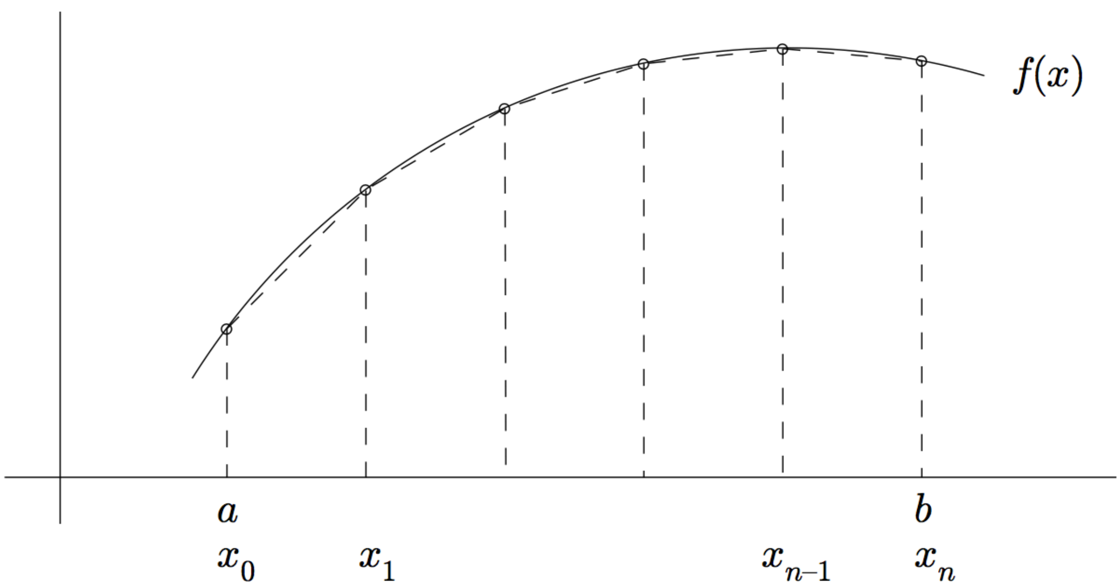
2. Interpolation at the Chebyshev points gives a good approximation on the entire interval.

3.4.1 Piecewise polynomial interpolation

We have seen that although unique, the polynomial $p_n(x)$ that interpolates a function f on $n+1$ data points may not be well conditioned. In situtations where we can control the interpolation locations, we can use Chebyshev points or other special points to minimize interpolation error. For cases where we do not have this flexibility, it is often more useful to use lower-degree polynomials, perhaps even lots of them. The interpolating polynomial $p_n(x)$ may not be a good approximation of $f(x)$ on the whole interval, so instead we may consider <u>piecewise linear interpolation</u>, denoted by $q(x)$.

Given $f(x),\ a \le x \le b,\quad a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b$

$$q(x) = \begin{cases} f[x_0] + f[x_0, x_1](x - x_0), & x_0 \le x < x_1 \\ \vdots & \vdots \\ f[x_i] + f[x_i, x_{i+1}](x - x_i), & x_i \le x < x_{i+1} \\ \vdots & \vdots \\ f[x_{n-1}] + f[x_{n-1}, x_n](x - x_{n-1}), & x_{n-1} \le x \le x_n \end{cases}$$



Notes:

- $q(x)$ is continuous, but not differentiable, at $x = x_i$
- error: $|f(x) - q(x)| \le \frac{1}{8} \max_{a \le x \le b} |f''(x)| \cdot \max_i |x_{i+1} - x_i|^2$: 2nd order accurace
- Piecewise interpolation is often refered to as a <u>local method</u>, as the function is locally approximatedly by a line at each grid point.

3.4 spline interpolation

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Cubic Spline Interpolation

Given a function f defined on $[a, b]$ and a set of nodes, $a = x_0 < x_1 < \cdots x_n = b$, a cubic spline interpolation S for f is a function that satisfies the following conditions

a) For each $i = 0, 1, \cdots, n-1$, $S(x)$ is a cubic polynomial denoted by $S_i(x)$ on the subinterval $[x_i, x_{i+1})$, i.e. $S(x) = S_i(x), \quad x_i \leq x < x_{i+1}$

b) $S_i(x_i) = f(x_i), S_i(x_{i+1}) = f(x_{i+1}), \quad i = 0, 1, \cdots, n-1$, i.e. $S(x_i) = f(x_i), i = 0, 1, \cdots, n$

c) $S_{i+1}(x_{i+1}) = S_i(x_{i+1})$ for $i = 0, 1, \cdots, n-2$: $S(x)$ is continuous at all interior points $x_1, \cdots, x_{n-1}$

d) $S'_{i+1}(x_{i+1}) = S'_i(x_{i+1})$ for $i = 0, 1, \cdots, n-2$: $S'(x)$ is continuous at all interior points $x_1, \cdots, x_{n-1}$

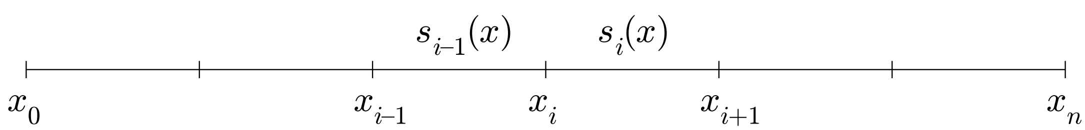
e) $S''_{i+1}(x_{i+1}) = S''_i(x_{i+1})$ for $i = 0, 1, \cdots, n-2$: $S''(x)$ is continuous at all interior points $x_1, \cdots, x_{n-1}$

f) One of the following sets of boundary conditions is satisfied

(i) $S''(x_0) = S''(x_n) = 0$: natural or free boundary

(ii) $S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n)$: clamped boundary

Question: why a)–e) not enough to determine $S(x)$?



$x_i \leq x \leq x_{i+1} \Rightarrow S(x) = S_i(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad i = 0 : n-1$

$n+1$ points $\Rightarrow n$ intervals $\Rightarrow 4n$ unknown coefficients

interpolation conditions $\Rightarrow 2n$ equations

continuity of $s'(x), s''(x)$ at interior points $\Rightarrow 2(n-1)$ equations

Hence we need $\underline{2}$ more conditions: nature bc is a popular choice

e.g. Construct a natural cubic spline that passes through the pints $(1, 2), (2, 3), (3, 5)$ i.e. $f(1) = 2, f(2) = 3, f(3) = 5$ and $x_0 = 1, x_1 = 2, x_2 = 3$

This spline consists of two cubics, the first for the interval $[1, 2]$, denoted

$$S_0 = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3$$

and the other for $[2,3]$, denoted

$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3$$

$a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1 \Rightarrow 8$ constants to be determined $\Rightarrow 8$ conditions

—— four splines must agree with the data at the nodes

$$2 = f(1) = a_0$$

$$3 = f(2) = a_0 + b_0 + c_0 + d_0$$

$$3 = f(2) = a_1$$

$$5 = f(3) = a_1 + b_1 + c_1 + d_1$$

—— two more come from the fact that

$$S_0'(2) = S_1'(2): \quad b_0 + 2c_0 + 3d_0 = b_1$$

$$S_0''(2) = S_1''(2): \quad 2c_0 + 6d_0 = 2c_1$$

—— The final two come from the nature boundary condition

$$S_0''(1) = 0: \quad 2c_0 = 0$$

$$S_1''(3) = 0: \quad 2c_1 + 6d_1 = 0$$

solve the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \end{pmatrix}_{8\times 8} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 2 \\ 5 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow S(x) = 2 + \tfrac{3}{4}(x-1) + \tfrac{1}{4}(x-1)^4, \quad x \in [1,2]$$

$$= 3 + \tfrac{3}{2}(x-2) + \tfrac{3}{4}(x-2)^2 - \tfrac{1}{4}(x-3)^3, \quad x \in [2,3]$$

In general, clamped bc more accurate because they include more information about the function f

<u>Q: how to find $s(x)$ in general?</u>

e.g. $-1 \le x \le 1,\ x_i = -1 + ih,\ h = \frac{2}{n},\ i = 0, ..., n$, uniform points

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Now we derive the equation for $S_i(x)$ on the interval $[x_i, x_{i+1})$, let $a_i = S''(x_i)$, $S''(x_i)$ continuous, i.e. $\lim_{x\to x_i^-} S''(x) = a_i = \lim_{x\to x_i^+} S''(x),\ 1 \le i \le n-1$: interior points.

—— Step 1: 2nd derivative condition. Because S'' is continuous at each interior points and $S_i(x)$ is a <u>cubic</u> polynomial

$\Rightarrow S_i''(x)$ is a <u>linear</u> function

let $S_i''(x_i) = a_i,\ \ S_i''(a_{i+1}) = a_{i+1}$, where a_i, a_{i+1} to be determined

a linear function (line) pass (x_i, a_i) and (x_{i+1}, a_{i+1}), and $x_{i+1} - x_i = h$

slope: $k = \dfrac{a_{i+1} - a_i}{x_{i+1} - i} = \dfrac{a_{i+1} - a_i}{h}$

the line equation is: $y = k(x - x_i) + a_i = \dfrac{a_{i+1} - a_i}{h}(x - x_i) + a_i$

$$= \frac{a_{i+1} - a_i}{h}(x - x_i) + a_i\frac{x_{i+1} - x_i}{h}$$

$$= \cdots = \frac{a_i}{h}(x_{i+1} - x) + \frac{a_{i+1}}{h}(x - x_i)$$

$$\Rightarrow S_i''(x) = \frac{a_i}{h}(x_{i+1} - x) + \frac{a_{i+1}}{h}(x - x_i)\ \ (*)$$

—— Step 2: interpolation

we have $(*)$ from step 1, integrate twice

$\Rightarrow S_i(x) = \dfrac{a_i}{6h}(x_{i+1} - x)^3 + \dfrac{a_{i+1}}{6h}(x - x_i)^3 + C(x - x_i) + D(x_{i+1} - x)$, where C, D are arbitrary constant (why this form? easy computation)

(to verify, differentiate $S_i(x)$ twice $\Rightarrow S_i''(x)$)

we then can use $S_i(x_i) = f_i,\ \ S_i(x_{i+1}) = f_{i+1} \Rightarrow C, D$

$$S_i(x_i) = \frac{a_i}{6h}(x_{i+1} - x_i)^3 + D(x_{i+1} - x_i) = f_i$$

$$S_i(x_{i+1}) = \frac{a_{i+1}}{6h}(x_{i+1} - x_i)^3 + C(x_{i+1} - x_i) = f_{i+1}$$

$$\Rightarrow C = \frac{f_{i+1}}{h} - \frac{a_{i+1}}{6}h,\ \ D = \frac{f_i}{h} - \frac{a_i}{6}h$$

$$\Rightarrow S_i(x) = \frac{a_i}{6h}(x_{i+1} - x)^3 + \frac{a_{i+1}}{6h}(x - x_i)^3 + (\frac{f_{i+1}}{h} - \frac{a_{i+1}}{6}h)(x - x_i) + (\frac{f_i}{h} -$$

$\frac{a_i}{6}h)(x_{i+1}-x)$

—— Step 3: 1st derivative conditions at interior points (to find $a_1, a_2, ..., a_{n-1}$)

$$S_i'(x) = \frac{-3a_i}{6h}(x_{i+1}-x)^2 + \frac{3a_{i+1}}{6h}(x-x_i)^2 + (\frac{f_{i+1}}{h} - \frac{a_{i+1}}{6}h) - (\frac{f_i}{h} - \frac{a_i}{6}h)$$

$$S_i'(x_i) = -\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1}h}{6}$$

$$S_i'(x_{i+1}) = \frac{a_{i+1}h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1}h}{6}$$

we require $S_{i-1}'(x_i) = S_i'(x_i)$ at all interior points $(i = 1, ..., n-1)$

$$S_i'(x_{i+1}) = \frac{1}{3}a_{i+1}h + \frac{f_{i+1}}{h} - \frac{f_i}{h} + \frac{a_i}{6}h$$

$$S_{i-1}'(x_i) = \frac{1}{3}a_i h + \frac{f_i}{h} - \frac{f_{i-1}}{h} + \frac{a_{i-1}}{6}h$$

$$= S_i'(x_i) = -\frac{a_i}{3}h - \frac{a_{i+1}}{6}h + \frac{f_{i+1}}{h} - \frac{f_i}{h}$$

$$\Rightarrow \frac{1}{3}a_i h + \frac{f_i}{h} - \frac{f_{i-1}}{h} + \frac{a_{i-1}}{6}h = -\frac{a_i}{3}h - \frac{a_{i+1}}{6}h + \frac{f_{i+1}}{h} - \frac{f_i}{h}$$

$$\Rightarrow \frac{a_{i-1}h}{6} + \left(\frac{a_i}{3} + \frac{a_i}{3}\right)h + \frac{a_{i+1}h}{6} = \frac{f_{i-1} - 2f_i + f_{i+1}}{h}, \quad i = 1 : n-1$$

$$\Rightarrow a_{i-1} + 4a_i + a_{i+1} = \tfrac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right), \quad i = 1 : n-1$$

<u>step 4</u> : apply BC

$S_0''(x_0) = a_0 = 0$, $S_{n-1}''(x_n) = a_n = 0$

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_{n-1} \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

A : symmetric , tridiagonal , positive definite

e.g. Determine whether $s(x)$ is a natural cubic spline.

a) $s(x) = \begin{cases} 0 & , 0 \le x \le 1 \\ x^3 - 3x^2 + 3x - 1 & , 1 \le x \le 2 \end{cases}$ b) $s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 \ , & -1 \le x \le 0 \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 \ , & 0 \le x \le 1 \end{cases}$

Need to check the following conditions:

(1) cubic polynominal on each subinterval
(2) $s''(a) = s''(b) = 0$
(3) $s_i(x_{i+1}) = s_{i+1}(x_i)$, $s_i'(x_{i+1}) = s_{i+1}'(x_i)$, $s_i''(x_{i+1}) = s_{i+1}''(x_i)$.

a) $s_0(x) = 0$, $0 \le x \le 1$; $s_1(x) = x^3 - 3x^2 + 3x - 1$, $1 \le x \le 2$
$s_0(x) = s_0'(x) = s_0''(x) = 0$, $s_1'(x) = 3(x-1)^2$, $s_2''(x) = 6(x-1)$
Since $s''(2) = s_2''(2) = 6 \neq 0$, it is not cubic spline.

b) $s_0(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$, $-1 \le x \le 0$; $s_1(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$, $0 \le x \le 1$
$s_0'(x) = -\frac{3}{2}x^2 - 3x$, $s_0''(x) = -3x - 3$; $s_1'(x) = \frac{3}{2}x^2 - 3x$, $s_1''(x) = 3x - 3$
Then $s_0''(-1) = 0$, $s_1''(1) = 0$, satisfy condition (2). $s_0(0) = 1 = s_1(0)$, $s_0'(0) = 0 = s_1'(0)$, $s_0''(0) = -3 = s_1''(0)$, satisfy condition (3). So it is a cubic spline. ok.

e.g.. Consider the natural cubic spline $s(x)$ satisfying $s(0) = 0, s(1) = 1, s(2) = 0$.

a) Find the value of $s''(1)$.

b) Sketch the graph of $s(x)$ on the interval $0 \le x \le 2$.

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $h = 1$. $f_0 = f(x_0) = 0$, $f_1 = f(x_1) = 1$, $f_2 = f(x_2) = 0$. Now go through the four steps in the notes.
(1) second derivative

$$\begin{aligned} s_0''(x) &= a_0\frac{x_1 - x}{h} + a_1\frac{x - x_0}{h} = a_0(x_1 - x) + a_1(x - x_0), \\ s_1''(x) &= a_1\frac{x_2 - x}{h} + a_2\frac{x - x_1}{h} = a_1(x_2 - x) + a_2(x - x_1), \end{aligned}$$

(2) interpolation

$$\begin{aligned} s_0(x) &= a_0\frac{(x_1 - x)^3}{6} + a_1\frac{(x - x_0)^3}{6} + (f_0 - \frac{a_0}{6})(x_1 - x) + (f_1 - \frac{a_1}{6})(x - x_0) \\ s_1(x) &= a_1\frac{(x_2 - x)^3}{6} + a_2\frac{(x - x_1)^3}{6} + (f_1 - \frac{a_1}{6})(x_2 - x) + (f_2 - \frac{a_2}{6})(x - x_1) \end{aligned}$$

(3) first derivative $a_0 + 4a_1 + a_2 = 6(f_0 - 2f_1 + f_2) = -12$
(4) apply BC $s_0''(x_0) = a_0 = 0$, $s_1''(x_1) = a_2 = 0$
Therefore, $s_0(x) = -\frac{1}{2}x^3 + \frac{3}{2}x$, $s_1(x) = -\frac{1}{2}(1-x)^3 + \frac{3}{2}(2-x)$.

3.5 Hermite interpolation

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Another way of handling Runge's phenomenon is to notice that although the interpolating polynomial $p_n(x)$ interpolates the data $p_n(x_i) = f(x_i)$ at each point x_i, the polynomial may intersect the graph of f at very steep angles. It would be better if the interpolating function $p(x)$ was parallel to $f(x)$ at the interpolating points; that its, we would like $p'(x_i) = f'(x_i)$. Cubic splines come close to achieving this, but don't quite get there.

Suppose we have values for $f(x_i)$ *and* $f'(x_i)$ on an interval $a \le x \le b$ with n subdivisions and $n+1$ points,

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \quad h = \frac{b-a}{n}, \quad x_i = a + ih, \quad i = 0, \ldots, n$$

we want to determine a polynomial $p(x)$ that interpolates the data and goes through each point (x_i, f_i) with slope $f'(x_i)$, that is, the polynomial p should satisfy

$$p(x_i) = f(x_i), \quad p'(x_i) = f'(x_i), \quad i = 0, \ldots, n$$

Note that the data include the function values and the function derivatives, each at $n+1$ points, for a total of $2n+2$ input values to the interpolation problem; hence we know p can have degree of at most $2n+1$.

Theorem 10. *Let the interval $a \le x \le b$ be divided into n subintervals by $n+1$ distinct points, and the function f and its first derivative f' be defined at each of these points. Then there exists a unique polynomial $p(x)$ of degree less than or equal to $2n+1$ such that*

$$p(x_i) = f(x_i), \qquad p'(x_i) = f'(x_i), \quad i = 0, \dots, n$$

Proof. Let $L_k(x)$ denote the Lagrange polynomial of degree $\le n$ centered about the kth data location,

$$L_k(x) = \prod_{\substack{i=0 \\ i \ne k}}^{n} \left(\frac{x - x_i}{x_k - x_i} \right).$$

Define the polynomials

$$\begin{aligned} H_k(x) &= \left(1 - 2L_k'(x_k)(x - x_k)\right) L_k^2(x) \\ \hat{H}_k(x) &= (x - x_k) L_k^2(x) \end{aligned}$$

Note that both of these polynomials have degree $\le 2n+1$. Also

$$\begin{aligned} H_k(x_i) &= \begin{cases} 1, & i = k \\ 0, & i \ne k \end{cases} \\ H_k'(x) &= -2L_k'(x_k)L_k^2(x) + 2\left(1 - 2L_k'(x_k)(x - x_k)\right) L_k(x) L_k'(x) \\ \Rightarrow H_k'(x_i) &= 0 \\ \hat{H}_k(x_i) &= 0 \\ \hat{H}_k'(x) &= L_k^2(x) + 2(x - x_k) L_k L_k'(x) \\ \Rightarrow \hat{H}_k'(x_i) &= \begin{cases} 1, & i = k \\ 0, & i \ne k \end{cases} \end{aligned}$$

These definitions and properties make it clear that H_k is associated with the function, and $\hat{H}_k$ is associated with the derivative at $x = x_k$.
Define the polynomial

$$p(x) = \sum_{k=0}^{n} f(x_k) H_k(x) + \sum_{k=0}^{n} f'(x_k) \hat{H}_k(x).$$

From the properties of H_k above, we have

$$p(x_i) = \sum_{k=0}^{n} f(x_k) H_k(x_i) + \sum_{k=0}^{n} f'(x_k) \hat{H}_k(x_i) = f(x_i)$$

$$p'(x_i) = \sum_{k=0}^{n} f(x_k) H_k'(x_i) + \sum_{k=0}^{n} f'(x_k) \hat{H}_k'(x_i) = f'(x_i)$$

for all $i = 0, \dots, n$. The polynomial p therefore interpolates all function and derivative values. Uniqueness follows from the fundamental theorem of algebra. □

As with our previous work with global interpolating polynomials, the Lagrange form is useful to establish existence, but cumbersome to use in practice. The Hermite interpolating polynomial is more efficiently computed and evaluated in Newton's form using a divided difference table. Before we start the table, we need to add another item to our understanding of divided differences:

Definition 11. Divided differences of the form $f[x_i, x_i]$ are understood to be $f[x_i, x_i] = f'(x_i)$. We justify this definition by considering the usual divided difference

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

in the limit $x_{i+1} \to x_i$. The divided difference in this limit is simply the definition of $f'(x_i)$,

$$\lim_{x_{i+1} \to x_i} f[x_i, x_{i+1}] = f[x_i, x_i] = \lim_{x_{i+1} \to x_i} \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}.$$

The divided difference table for Hermite interpolation is augmented from the previous version to account for the addition of the derivative data :

x	$f(x)$		
x_0	$f[x_0]$		
		f$[x_0, x_0]$	
x_0	$f[x_0]$		$f[x_0, x_0, x_1]$
		f$[x_0, x_1]$	
x_1	$f[x_1]$		$f[x_0, x_1, x_1]$
		f$[x_1, x_1]$	
x_1	$f[x_1]$		$f[x_1, x_1, x_2]$
		f$[x_1, x_2]$	
x_2	$f[x_2]$		$f[x_1, x_2, x_2]$
		f$[x_2, x_2]$	
x_2	$f[x_2]$		$f[x_2, x_2, x_3]$
		f$[x_2, x_3]$	
x_3	$f[x_3]$		$f[x_2, x_3, x_3]$
		f$[x_3, x_3]$	
x_3	$f[x_3]$		$f[x_3, x_3, x_4]$
		f$[x_3, x_4]$	
x_4	$f[x_4]$		$f[x_3, x_4, x_4]$
		f$[x_4, x_4]$	$\vdots$
x_4	$f[x_4]$	$\vdots$	
$\vdots$	$\vdots$		

<u>Example 11:</u> Hermite interpolation of $f(x) = xe^{-x}$ on $0 \le x \le 4$ with $n = 2$

$\Rightarrow x_0 = 0,\ x_1 = 2,\ x_2 = 4$

$f'(x) = (1 - x)e^{-x}$

x	$f(x)$					
0	0					
		$f'(0) = 1$				
0	0		$(e^{-2} - 1)/2$			
		e^{-2}		$(-3e^{-2} + 1)/4$		
2	$2e^{-2}$		$-e^{-2}$		$(e^{-4} + 4e^{-2} - 1)/16$	
		$f'(2) = -e^{-2}$		$(e^{-4} + e^{-2})/4$		$(-9e^{-4} - 4e^{-2} + 1)/64$
2	$2e^{-2}$		e^{-4}		$-e^{-4}/2$	
		$2e^{-4} - e^{-2}$		$(-7e^{-4} + e^{-2})/4$		
4	$4e^{-4}$		$(-5e^{-4} + e^{-2})/2$			
		$f'(4) = -3e^{-4}$				
4	$4e^{-4}$					

To write the polynomial, we introduce the sequence $\{z_k\}$ where $z_k = x_{\lfloor k/2 \rfloor}$, $k = 0, \ldots, 2n+1$:

$$\{z_k\} = \{x_0, x_0, x_1, x_1, x_2, x_2, \ldots, x_{n-1} x_{n-1}, x_n, x_n\}$$

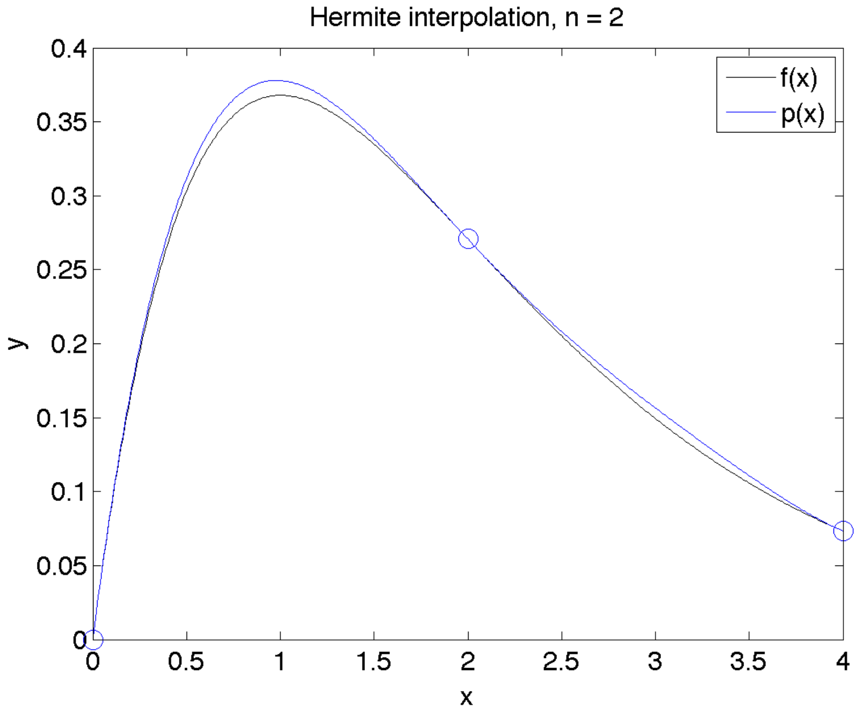
$$p(x) = \sum_{k=0}^{2n+1} f[z_0, z_1, \ldots, z_k] \left(\prod_{i=0}^{k-1} (x - z_i) \right)$$

The terms of this sum are

$$\begin{array}{lll}
k=0: & f[z_0] & = f(x_0) \\
k=1: & f[z_0, z_1](x - z_0) & = f'(x_0)(x - x_0) \\
k=2: & f[z_0, z_1, z_2](x - z_0)(x - z_1) & = f[x_0, x_0, x_1](x - x_0)^2 \\
k=3: & f[z_0, z_1, z_2, z_3](x - z_0)(x - z_1)(x - z_2) & = f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1) \\
k=4: & f[z_0, \ldots, z_4](x - z_0)(x - z_1)(x - z_2)(x - z_3) & = f[x_0, x_0, x_1, x_1, x_2](x - x_0)^2(x - x_1)^2 \\
k=5: & f[z_0, \ldots, z_5](x - z_0)(x - z_1)(x - z_2)(x - z_3)(x - z_4) & = f[x_0, x_0, x_1, x_1, x_2, x_2](x - x_0)^2(x - x_1)^2(x - x_2)
\end{array}$$

Note that the coefficients are conveniently the top entry in each column of the divided difference table. Plugging those values in, along with $x_0 = 0,\ x_1 = 2,\ x_2 = 4$, we find

$$p(x) = x + \left(\frac{e^{-2} - 1}{2} \right) x^2 - \left(\frac{3e^{-2} - 1}{4} \right) x^2(x-2) + \left(\frac{e^{-4} + 4e^{-2} - 1}{16} \right) x^2(x-2)^2 - \left(\frac{9e^{-4} + 4e^{-2} - 1}{64} \right) x^2(x-2)^2(x-4)$$



△

Notes:

- Hermite interpolation can be very accurate, but the $2n+1$ data points can be expensive to use if n is large.
- Can we use this idea more efficiently?

4.1 Piecewise cubic Hermite interpolation

Definition 12. The Hermite cubic interpolant of $f(x)$ on the interval $a \le x \le b$ with n subintervals

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

is a function s that satisfies

1. $s(x)$ coincides with a cubic polynomial $s_i(x)$ on each subinterval
2. s interpolates both $f(x)$ and $f'(x)$ at each data point
3. $s(x)$ and $s'(x)$ are continuous on $a \le x \le b$.

Note that there is no continuity condition imposed on $s''(x)$, hence cubic Hermite splines are not as smooth as natural cubic splines, but computing them requires less work.

Definition 13. The shape functions $\phi(\xi)$ and $\psi(\xi)$ associated with cubic Hermite interpolation are

$$\phi(\xi) = (1+2\xi)(1-\xi)^2$$
$$\psi(\xi) = \xi(1-\xi)^2$$

where the variable ξ is defined on each subinterval as $\xi_j = \dfrac{x - x_j}{x_{j+1} - x_j}$.

Definition 14. Given $f(x_i)$ and $f'(x_i)$ for $i = 0, \dots, n$, the cubic Hermite interpolant is defined on each subinterval of $a \le x \le b$ to be

$$s_j(x) = f(x_{j+1}) + \phi(\xi_j)\left(f(x_j) - f(x_{j+1})\right) + (x_{j+1} - x_j)\left(\psi(\xi_j)f'(x_j) - \psi(1-\xi_j)f'(x_{j+1})\right)$$

so that

$$s(x) = s_j(x) \text{ for } x_j \le x \le x_{j+1}, \qquad j = 0, \dots, n-1$$

Notes:

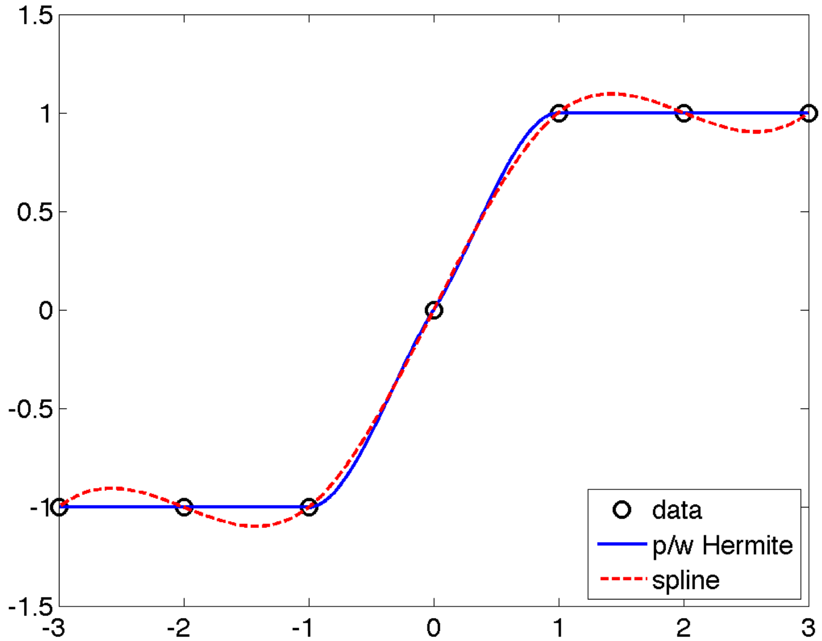
- Unlike cubic splines, no linear system of equations has to be solved
- Only three function evaluations : $\phi(\xi_j), \psi(\xi_j)$, and $\psi(1-\xi_j)$ are required to evaluate $s_j(x)$.
- Like any local method, still have to locate the interval j that contains the evaluation point to begin
- If the function itself is not differentiable, it is better to use the less-smooth cubic Hermite interpolation than cubic splines
- If you know the function you're interpolating is smooth, splines will provide better accuracy than cubic Hermite interpolation

Question 4: It's great that Hermite interpolation is accurate and that piecewise cubic Hermite interpolation is so fast, but what if we don't have data about $f'(x)$? Can we still use these ideas?

Example 12: Compare Matlab's piecewise cubic spline interpolation to its piecewise cubic Hermite interpolation for the data given in the following table.

x	y
-3	-1
-2	-1
-1	-1
0	0
1	1
2	1
3	1

△




```
clear;
%% Cubic Hermite interpolation vs. cubic spline interpolation

x = -3:3;                          % data locations
y = [-1,-1,-1, 0, 1,1,1];          % data values

xe = -3:0.01:3;                    % evaluation domain

sHermite = pchip(x,y,xe);          % cubic Hermite interpolation
sSpline = spline(x,y,xe);          % cubic spline interpolation
% Note: read Matlab's documentation to learn how the derivatives are
% estimated for the Hermite interpolation.  It's more involved than a
% simple finite difference scheme -- they use limiters (Math 572).

figure(1);clf;
plot(x,y,'ko',xe,sHermite,'b-',xe,sSpline,'r--','MarkerSize',10,'LineWidth',2);
set(gca,'FontSize',16);
legend('data','p/w Hermite','spline','Location','SouthEast');

saveas(1,'splineVsHermite.png')
```