

2. (i) Given $\alpha_1, \alpha_2, \dots, \alpha_{n-1} \in \mathcal{A}$, the set

$$[\alpha_1, \alpha_2, \dots, \alpha_{n-1}] = \{\omega \in \mathcal{A}^{\mathbb{N}} : a_j = \alpha_j, 0 \leq j \leq n-1, \text{ where } \omega = (a_0, a_1, a_2, \dots)\},$$

is called a *cylinder set*. Prove that cylinder sets are both open and closed (called *clopen*), and in fact if $\alpha \in C$

$$C = \{\omega \in \mathcal{A}^{\mathbb{N}} : d(\alpha, \omega) < 1/2^{n-1}\} = \{\omega \in \mathcal{A}^{\mathbb{N}} : d(\alpha, \omega) \leq 1/2^n\}.$$

(ii) Show that any clopen set is the finite union of cylinder sets.

2. Let $f_n = \theta^n(0)$, $n \geq 0$ and $f_{-1} = 1$, where $u = u_0 u_1 \dots$ is the Fibonacci sequence generated by the substitution $\theta(0) = 01$, $\theta(1) = 0$.

(a) Show that $f_4 = \epsilon(010)(010)10$, and $f_5 = 0(10010)(10010)01$ and now use induction on the length of a word to show that for any word w

$$\theta(\tilde{w})0 = 0\theta(\tilde{w}).$$

(where $\tilde{w}$ denotes the reflection of the word w , so if $w = a_0 a_1 \dots a_n$, then $\tilde{w} = a_n a_{n-1} \dots a_0$).

(b) Use (a) to prove the identity

$$f_{n+2} = v_n \tilde{f}_n \tilde{f}_n t_n, \quad n \geq 2,$$

where $v_2 = \epsilon$, and for $n \geq 3$

$$v_n = f_{n-3} \cdots f_1 f_0, \quad \text{and} \quad t_n = \begin{cases} 01, & \text{if } n \text{ is odd} \\ 10, & \text{otherwise} \end{cases}.$$

(Hint: Show that $\theta(\tilde{f}_n t_n) = 0 \tilde{f}_{n+1} t_{n+1}$, then use $\theta(v_n)0 = v_{n+1}$).

7. A substitution ζ is defined on $\mathcal{A} = \{0, 1, 2, 3\}$ by

$$\zeta(0) = 02; \quad \zeta(1) = 32; \quad \zeta(2) = 01; \quad \zeta(3) = 31.$$

Let ε_n be the n th term in Rudin-Shapiro sequence. If $\tau : \mathcal{A} \rightarrow \{-1, 1\}$ is defined by $\tau(0) = 1 = \tau(2)$ and $\tau(1) = -1 = \tau(3)$, show that $\tau(\zeta^\infty(0)) = \varepsilon$.

3. Let T_α be an irrational rotation on $[0, 1)$, $0 < \alpha < 1$. Show that $T_\alpha^n(0) \in [1 - \alpha, 1)$ if and only if $\lfloor (n + 1)\alpha \rfloor - \lfloor n\alpha \rfloor = 1$, i.e., $\{n\alpha\} \in [1 - \alpha, 1)$ if and only if $f_\alpha(n) = 1$.

4. A substitution θ is defined on $S = \{0, 1, 2\}$ by

$$\theta(0) = 010, \quad \theta(1) = 121, \quad \theta(2) = 202.$$

In this case the sequence $\theta^n(0)$ converges to a sequence

$$x = 010121010121202121 \dots$$

If (x_n) is the n th term of the resulting sequence, prove that

$$x_{3n} = x_n, \quad x_{3n+1} = x_n + 1, \quad x_{3n+2} = x_n, \quad n \geq 0,$$

(addition being modulo three).

6. Show that the Rauzy graph Γ_n is always connected, i.e., given any two vertices U and V , there are edges that connect U and V .