QUAN 203 - 2021

Tutorial 6: try, scan and upload these exercises to Blackboard before attending your tutorial in Week 10. Those submitted by the notified deadline will be scored and contribute towards the Tutorial Assignment component of your final mark.

1. Let X_1 and X_2 be independent random variables with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and moment generating functions $M_1(t)$ and $M_2(t)$ respectively. Also, let $Y = X_1 X_2$.

   (a) Using results on expectation for products of independent random variables, find $E(Y)$ and $var(Y)$ in terms of μ_1, μ_2, σ_1^2 and σ_2^2.

   (b) Noting that $M_Y(t) = M_1(t) \times M_2(t)$, use the product rule and standard results on moment generating functions to confirm $E(Y)$ and $var(Y)$ above.

2. The pdf of a standard bivariate normal random variable (X, Y) is

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right\}$$

   Prove that when $\rho = corr(X,Y) = 0$, X and Y are independent $N(0,1)$ random variables.

3. Let $Y = (n-1)S^2/\sigma^2$ where $Y \sim \chi^2_{n-1}$ so that S^2 is the sample variance of an iid sample of $N(\mu, \sigma^2)$ random variables. Noting that a χ^2_ν random variable has mgf $M(t) = (1-2t)^{-\nu/2}$, show that the mean and variance of Y are $n-1$ and $2(n-1)$ respectively. Hence or otherwise, find the mean and variance of S^2.

4. Suppose $X_1, \ldots, X_n$ are iid random variables with mean μ and variance σ^2, but with unknown distribution. Show that the sample variance

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1} = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n-1}$$

   is an unbiased estimator for σ^2, i.e. show, using results on the linearity of expectation (e.g. $E(a + bX) = a + bE(X)$), that $E(S^2) = \sigma^2$.

   NB: when the X_i are not normal, $(n-1)S^2/\sigma^2$ is not χ^2_{n-1}.

5. According to wikipedia.org, the moments of the t random variable, X, with ν degrees of freedom are given by:

$$E(X^k) = \begin{cases} 0 & k \text{ odd, } 0 < k < \nu \\ \frac{1}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\left[\Gamma(\frac{k+1}{2})\Gamma(\frac{\nu-k}{2})\nu^{\frac{k}{2}}\right] & k \text{ even, } 0 < k < \nu \\ \text{undefined} & k \text{ odd, } k \geq \nu \\ \infty & k \text{ even, } k \geq \nu. \end{cases}$$

   Noting $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, derive $\sigma_X^2 = E\{(X-\mu)^2\}$ and $E\{(X-\mu)^4\}/\sigma_X^4$, i.e. the variance and kurtosis. Note any restrictions on these.