

## HOMEWORK FOR WEEK OF JAN 25

All section numbers refer to Ablowitz-Fokas.

- (1) Problems 1, 2, 3, 4, 7, 8, in Section 2.4. In Problem 3,  $C$  is the square with diagonal corners at  $-1 - i$  and  $1 + i$ . Remember that by convention, closed contours are oriented counterclockwise unless stated otherwise.
- (2) Problems 1, 2, 3 in Section 2.5

### SELECTED SOLUTIONS (UPDATED FEB 7)

- Problem 2.4.1b: Parametrise  $C$  as  $z = e^{it}$ ,  $0 \leq t \leq 2\pi$ , and compute

$$\int_C \bar{z}^2 dz = \int_0^{2\pi} (e^{-2it}) i e^{it} dt = \int_0^{2\pi} i e^{-it} dt = 0.$$

- Problem 2.4.2d: Use the same parametrisation as above:

$$\int_C |z|^2 dz = \int_0^{2\pi} i e^{it} dt = 0.$$

- Problem 2.4.4a: **Note:** the textbook uses the nonstandard convention of defining the principal branch of  $\log z$  to have imaginary part in the range  $[0, 2\pi)$ , rather than  $(-\pi, \pi]$ . With this convention in mind (although in this case, both conventions yield the same answer), we have

$$\log(x) = \begin{cases} \ln |x|, & x > 0 \\ \ln |x| + i\pi, & x < 0 \end{cases}$$

Thus

$$\begin{aligned} \int_{-1}^1 \log x dx &= \int_{-1}^0 \ln |x| + i\pi dx + \int_0^1 \ln |x| dx \\ &= 2 \int_0^1 \ln x dx + i\pi \\ &= \lim_{a \rightarrow 0^+} 2[x \ln x - x]_a^1 + i\pi \\ &= -2 + i\pi. \end{aligned}$$

- Problem 2.4.7: For  $z \in C$ ,  $|z| = R > a$ , so by the (reverse) triangle inequality  $|z^2 + a^2| \geq |z|^2 - a^2 = R^2 - a^2 \Rightarrow |f(z)| \leq \frac{1}{R^2 - a^2}$ . Since  $C$  has length  $\pi R$ , we obtain the bound

$$\left| \int_C f(z) dz \right| \leq \frac{\pi R}{R^2 - a^2}.$$

- Problem 2.5.2a: Decompose

$$\frac{1}{z(z-2)} = \frac{1}{2} \left[ \frac{1}{z-2} - \frac{1}{z} \right],$$

and write

$$\int_C \frac{1}{z(z-2)} dz = \frac{1}{2} \int_C \frac{1}{z-2} dz - \frac{1}{2} \int_C \frac{1}{z} dz.$$

The first integral equals zero by Cauchy's integral theorem. For the second integral, parametrise  $z = e^{it}$ ,  $dz = ie^{it} dt$ ,  $0 \leq t \leq 2\pi$ , and we obtain

$$\int_C \frac{1}{z(z-2)} dz = -\frac{1}{2} \int_0^{2\pi} i dt = -i\pi.$$