

HOMEWORK FOR WEEK OF FEB 1

All section numbers refer to Ablowitz-Fokas.

- (1) Problems 1, 2, 6, 7 in Section 2.6
- (2) In this exercise, you will evaluate the definite integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$$

using contour integration and Cauchy's integral formula.

- (a) Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx.$$

- (b) For each $R > 10$, let C_R be the following semicircular contour:

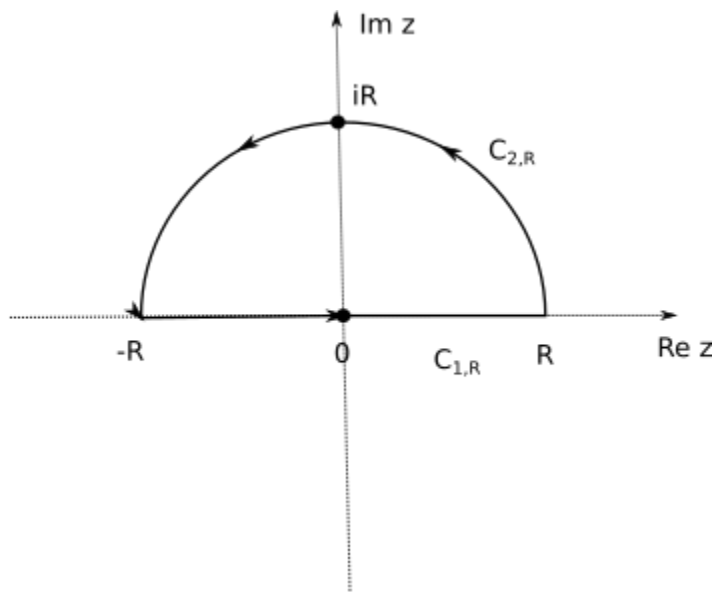


FIGURE 1. The contour $C_R = C_{1,R} + C_{2,R}$

Here, $C_{1,R} = [-R, R]$ traversed from left to right, and $C_{2,R}$ is the upper semicircle with radius R centered at 0.

Evaluate

$$\int_{C_R} \frac{e^{iz}}{1+z^2} dz.$$

(c) Note that

$$\int_{C_R} \frac{e^{iz}}{1+z^2} dz = \int_{C_{1,R}} \frac{e^{iz}}{1+z^2} dz + \int_{C_{2,R}} \frac{e^{iz}}{1+z^2} dz.$$

Using an ML bound, evaluate the limit

$$\lim_{R \rightarrow \infty} \int_{C_{2,R}} \frac{e^{iz}}{1+z^2} dz.$$

It will help here to recall the triangle inequality

$$||z| - |w|| \leq |z + w| \leq |z| + |w|, \quad z, w \in \mathbb{C}.$$

(d) Take the limit as $R \rightarrow \infty$ on both sides of the equation in part (c), and deduce the value of

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx.$$

SELECTED SOLUTIONS

- Problem 2.6.1b:

$$\int_{|z|=1} \frac{dz}{(2z-1)^2}$$

We write the integrand in the form of Cauchy's Integral Formula (for derivatives):

$$\frac{1}{(2z-1)^2} = \frac{1}{4} \frac{1}{(z-\frac{1}{2})^2} = \frac{f(z)}{(z-\frac{1}{2})^2},$$

where $f(z) = \frac{1}{4}$. By Cauchy's integral formula,

$$\int_{|z|=1} \frac{f(z)}{(z-\frac{1}{2})^2} dz = 2\pi i f'(\frac{1}{2}) = 0.$$

- Problem 2.6.2b:

$$\int_C \frac{e^z}{z - \frac{\pi i}{4}a} dz,$$

where C is the (counterclockwise-oriented) square with vertices $\pm R \pm iR$, $R > a$. Noting that C encloses the point $\frac{\pi i}{4}a$, we apply Cauchy's integral formula setting $f(z) = e^z$:

$$\int_C \frac{f(z)}{z - \frac{\pi i}{4}a} dz = 2\pi i f(\frac{\pi i a}{4}) = 2\pi i e^{\frac{\pi i a}{4}}.$$

- Problem 2.6.7. Observe first of all that the inequality $|f(z)| \leq C|z|$ implies that $f(0) = 0$, hence that

$$f'(0) = \lim_{z \rightarrow 0} \frac{f(z)}{z}.$$

Now consider the function

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$$

Then g is continuous at $z = 0$ and analytic at every $z \neq 0$. We claim that $\int_T g(z) dz = 0$ along any triangle T , which by Morera would imply that g is analytic.

There are three cases to consider:

- (1) T does not enclose $z = 0$. Then $\int_T g(z) dz = 0$ by the standard Cauchy Integral Theorem.
- (2) T passes through $z = 0$. Then by continuity, $\int_T g(z) dz = 0 = \lim_{\varepsilon \rightarrow 0} \int_{T_\varepsilon} g(z) dz$, where T_ε is T shifted by distance ε to avoid $z = 0$.
- (3) T encloses $z = 0$. By the deformation of contours technique (see figure),

$$\int_T g(z) dz = \int_{|z|=\varepsilon} g(z) dz$$

for all ε small enough that the circle $|z| = \varepsilon$ is enclosed by T ; this uses crucially

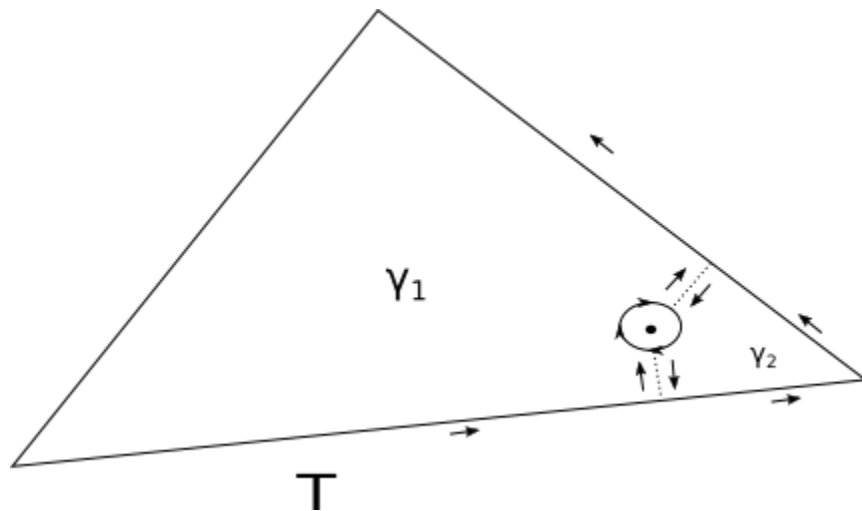


FIGURE 2. Contour deformation: introduce the two dotted segments and write $\int_T g dz - \int_{|z|=\varepsilon} g dz = \int_{\gamma_1} g dz + \int_{\gamma_2} g dz = 0$.

the fact that g is analytic everywhere except possibly at $z = 0$.

Since g is continuous on a neighbourhood of 0, it is bounded: $|g(z)| \leq C$ for all z . Hence

$$\left| \int_{T_\varepsilon} g(z) dz \right| \leq 2\pi\varepsilon C.$$

Taking $\varepsilon \rightarrow 0$ on both sides of the above equality, we conclude that

$$\int_T g(z) dz = 0.$$

Having established that g is entire and bounded, Liouville implies that $g(z) \equiv A$ for some constant A . Thus $f(z) = Az$.