

5 | Induction

5.1 Mathematical Induction

Example 5.1.1 (The Babies Example).

Baby #1: "can take one step". This baby ~~can't~~ necessarily walk.

Baby #2: "can add a step to a previously taken step". This baby may not necessarily be able to take the first step and therefore may not be able to walk.

Baby #3: "can take the first step AND can add a step to a previously taken step": can definitely walk.

Example 5.1.2 (The Ladder Example).



this is called the inductive hypothesis.

Theorem 5.1.1 (Principle of Mathematical Induction). To prove a statement a statement $P(n)$ is true for every integer $n \geq 1$, it suffices to show the following

(a) $P(1)$ is true. [basis step]

(b) The implication $P(k) \rightarrow P(k+1)$ is true, where $k \geq 1$ is an arbitrary integer. [inductive step]

Note. We may change the number 1 (in $n \geq 1$, $P(1)$, and $k \geq 1$) to some other integer m in the Principle of Mathematical Induction and theorem will remain true.

Note. In the Principle of Mathematical Induction, both steps (the basis step and the inductive step) are crucial!

T T T
 $P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \dots$ 80

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(n = 1, 2, 3, ...)

Example 5.1.4. Consider the recursion $a_{n+1} = a_n + \frac{1}{2^{n+1}}$ with $a_1 = \frac{1}{2}$.

- (a) Find a_2, a_3 , and a_4 .
 (b) Based on your answers in (a), conjecture an explicit formula for a_n .
 (c) Use mathematical induction to prove your conjecture.

$$(n=1) \rightarrow a_2 = a_1 + \frac{1}{2^{1+1}} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$(n=2) \rightarrow a_3 = a_2 + \frac{1}{2^{2+1}} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$(n=3) \rightarrow a_4 = a_3 + \frac{1}{2^{3+1}} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$$

(b) Conjecture:

$$\forall n \in \mathbb{N} : a_n = \frac{2^n - 1}{2^n}$$

(c) Let $P(n)$ be the statement " $a_n = \frac{2^n - 1}{2^n}$ ".

Basis step: $P(1)$: " $a_1 = \frac{2^1 - 1}{2^1}$ " is true because both sides are equal to $\frac{1}{2}$.

Inductive step: Assume $P(k)$ is true. That means

$$a_k = \frac{2^k - 1}{2^k} \quad (\text{inductive hypothesis})$$

Then

$$a_{k+1} = a_k + \frac{1}{2^{k+1}}$$

by inductive hypothesis $\downarrow$

$$= \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}}$$

$$2^{k+1} = 2^k \cdot 2^1 = 2^k \cdot 2$$

$$= \frac{2(2^k - 1) + 1}{2^{k+1}} = \frac{2^{k+1} - 2 + 1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$

$\therefore P(k+1)$ must be true

Example 5.1.5. Let $n \geq 7$ be an integer. Prove that $3^n < n!$.

Let $P(n)$ be the statement " $3^n < n!$ ".

Basis step $P(7)$: " $3^7 < 7!$ " is true because $3^7 = 2187$
 $7! = 5040$

Inductive step: Assume $P(k)$ is true for some $k \geq 7$.

That $3^k < k!$ ($k \geq 7$) (inductive hypothesis)

Then we have $3^{k+1} = 3 \cdot 3^k < (k+1) \cdot 3^k < (k+1) \cdot k! = (k+1)!$
 (because $3 < k+1$ when $7 \leq k$) by inductive hypothesis

$$\therefore 3^{k+1} < (k+1)!$$

$\therefore P(k+1)$ must be true.

$\therefore$ By induction $P(n)$ must be true for all $n \geq 7$.

Example 5.1.6. Let n be a non-negative integer. Prove that 3 divides $n^3 - n$.

$$n \geq 0$$

call this statement $P(n)$

① Proof (By induction). Basis step $P(0)$: " 3 divides $0^3 - 0$ "

is true because $3 \cdot \boxed{0} = 0^3 - 0$.

Inductive step: Assume $P(k)$ is true ($k \geq 0$)

That is assume $3 \mid k^3 - k$. $\therefore \exists m \in \mathbb{Z} : 3m = k^3 - k$. (inductive hypothesis)

$$\begin{aligned} \text{Then } (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 3k - k \\ &= 3k^2 + 3k + (k^3 - k) = 3k^2 + 3k + 3m \\ &= 3(k^2 + k + m) \end{aligned}$$

by ind. hyp.

$\sim$ an integer

$$\therefore 3 \mid (k+1)^3 - (k+1)$$

$\therefore P(k+1)$ is true.

Example 5.1.7. Let n be a positive integer. Prove that $4^{n+1} + 5^{2n-1} \equiv 0 \pmod{21}$.
 call this $P(n)$

Basis step : $P(1) : 4^{1+1} + 5^{2-1} \equiv 16 + 5 \equiv 21 \equiv 0 \pmod{21}$

$\therefore P(1)$ is true.

Inductive step: Assume $P(K) : 4^{K+1} + 5^{2K-1} \equiv 0 \pmod{21}$ is true for some $K \geq 1$. inductive hypothesis

We need to prove $4^{(K+1)+1} + 5^{2(K+1)-1} \equiv 0 \pmod{21}$.

$$\begin{aligned}
 4^{(K+1)+1} + 5^{2(K+1)-1} &\equiv 4^{K+1} \cdot 4 + 5^{2K-1} \cdot 5^2 \\
 &\equiv 4 \cdot 4^{K+1} + 25 \cdot 5^{2K-1} \\
 &\equiv 4 \cdot 4^{K+1} + 4 \cdot 5^{2K-1} + 21 \cdot 5^{2K-1} \\
 &\equiv 4 \cdot (4^{K+1} + 5^{2K-1}) + 21 \cdot 5^{2K-1} \\
 &\equiv 4 \cdot 0 + 0 \cdot 5^{2K-1} \pmod{21}
 \end{aligned}$$

(25 = 4 + 21)

by inductive hypothesis

$\therefore P(K+1)$ is true. $\therefore$ By induction $P(n)$ is true for all $n \geq 1$.

Example 5.1.8. What is wrong with this "proof" that all horses are the same colour?

Let $P(n)$ be the statement that all horses in a set of n horses are the same colour.

Base Step: $P(1)$ is certainly true, since with just one horse, all horses have the same colour.

Inductive Step: Assume $P(k)$, which is the statement that k horses all have the same colour (inductive hypothesis). Given a set of $k+1$ horses $\{h_1, h_2, \dots, h_{k+1}\}$, we can exclude the last horse in the set and apply the inductive hypothesis just to the first k horses $\{h_1, \dots, h_k\}$, deducing that they all have the same colour. Similarly, we can conclude that the last k horses $\{h_2, \dots, h_{k+1}\}$ all have the same colour. But now the "middle" horses $\{h_2, \dots, h_k\}$ (i.e., all but the first and the last) belong to both of these sets, so they have the same colour as horse h_1 and horse h_{k+1} . It follows, therefore, that all $k+1$ horses have the same colour. Thus, by the principle of induction, all horses have the same colour.

The proof of the implication $P(k) \rightarrow P(k+1)$ is true

only works when $k \geq 2$.

For $k=1$ there will be no "middle" horses and therefore the proof of " $P(1) \rightarrow P(2)$ is true" fails.

$$P(1) \xrightarrow{\text{X}} P(2) \xrightarrow{\text{T}} P(3) \xrightarrow{\text{T}} \dots$$

5.2 Strong Induction

Theorem 5.2.1 (Principle of Strong Induction). To prove a statement a statement $P(n)$ is true for every integer $n \geq 1$, it suffices to show the following

- (a) $P(1)$ is true.
 (b) The implication $\underline{P(1) \wedge P(2) \wedge \dots \wedge P(k)} \rightarrow P(k+1)$ is true, where $k \geq 1$ is an arbitrary integer.

Stronger inductive hypothesis
 (we may assume all of $P(1), \dots, P(k)$ are true before proving $P(k+1)$ is true)

Example 5.2.1. Prove that every integer $n \geq 1$ can be written as a sum of distinct powers of 2 (e.g. 41 can be written as $2^0 + 2^3 + 2^5$). (This means that every integer must have binary representation.)

$P(n)$: "n can be written as sum of distinct powers of 2"

Proof by strong induction :

Basis step: $P(1)$ is true because $1 = 2^0$

Inductive step: Assume $P(1), P(2), \dots, P(k)$ are all true for some $k \geq 1$.
 Now, we need to prove $P(k+1)$ must be true.

Case 1 If k is even : Because $P(k)$ is true, we can write

$$k = 2^{a_1} + 2^{a_2} + \dots + 2^{a_e} \text{ with } a_1, \dots, a_e \text{ distinct}$$

Since k is even $a_1, \dots, a_e \neq 0$

$$\therefore k+1 = 2^{a_1} + \dots + 2^{a_e} + 2^0 \text{ with } 0, a_1, \dots, a_e \text{ distinct}$$

$\therefore P(k+1)$ must be true.

Case 2 If k is odd : Since k is odd $\therefore k+1$ is even $\therefore \frac{k+1}{2} \in \mathbb{Z}$

also $\frac{k+1}{2} \leq k$ (because $k \geq 1$) $\therefore$ By inductive hypothesis $P(\frac{k+1}{2})$ must be true : $\frac{k+1}{2} = 2^{b_1} + 2^{b_2} + \dots + 2^{b_s}$ ($b_1, \dots, b_s$: distinct)

$$\text{Then } k+1 = 2(2^{b_1} + \dots + 2^{b_s}) = 2^{b_1+1} + 2^{b_2+1} + \dots + 2^{b_s+1} \text{ } (b_1+1, \dots, b_s+1 : \text{distinct})$$

$\therefore P(k+1)$ must be true.