

Fourier Approximation

Problem C: **Given:** 1) A function $g \in \mathcal{C}[a, b]$, and
2) “simple” functions $f_1, \dots, f_k \in \mathcal{C}[a, b]$.

Find: a linear combination

$$f_0 = a_1 f_1 + \dots + a_k f_k$$

which **best approximates** the function g in the L^2 -**norm**; in other words, find f_0 as above such that

$$\int_a^b (f_0 - g)^2 dx \leq \int_a^b (f - g)^2,$$

for all functions $f = a'_1 f_1 + \dots + a'_n f_n$.

Solution: **Step 1:** Apply the **Gram-Schmidt Method** to the functions $f_1, \dots, f_k$ by defining the **dot product** of two functions $f, g \in \mathcal{C}[a, b]$ to be

$$(f \cdot g) = \int_a^b f(x)g(x)dx.$$

This gives us **orthogonal functions** $h_1, \dots, h_k$ which are certain linear combinations of $f_1, \dots, f_k$.

Step 2: Use the **projection formula** of **Theorem 9:**

$$f_0 = P_V(g) = \sum_{i=1}^k \frac{(g \cdot h_i)}{(h_i \cdot h_i)} h_i$$