

Math 211

Assignment 12

Due 31 January 2020

Reminder: Midterm Test on Wednesday, 12 February, 9:30–10:30.

- [5] 1. (a) Let $\vec{y} = (3, 1, 2, 2)^t$ and consider the 4×2 matrix $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}^t$. Find the unique vector $\vec{w} \in \mathbb{R}^2$ such that $\|A\vec{w} - \vec{y}\|$ is minimal.
- (b) Similarly, find the unique vector $\vec{w} \in \mathbb{R}^3$ such that $\|B\vec{w} - \vec{z}\|$ is minimal when $B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}^t$ and $\vec{z} = (2, 0, 3, 1, 2)^t$.
- [2] 2. Among all the points in the linear subspace $L = \langle \vec{v}_1, \vec{v}_2 \rangle$ of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = (1, 1, 1, 1)^t$ and $\vec{v}_2 = (1, 0, 1, 0)^t$, find the one that is closest to the vector $\vec{y} = (1, 1, 1, 0)^t$. Moreover, find the distance $d(\vec{y}, L) = \min\{d(\vec{y}, \vec{v}) : \vec{v} \in L\}$ from $\vec{y}$ to L .
- [2] 3. Let A be an $n \times k$ matrix and $\vec{y} \in \mathbb{R}^n$ be a vector. Prove that $A^t \vec{y} = \vec{0}$ if and only if $(\vec{y} \cdot A\vec{w}) = 0$, for all $\vec{w} \in \mathbb{R}^k$.
- [4] 4. Let $\vec{v}_1 = (1, 1, 0, 0)^t$, $\vec{v}_2 = (1, 1, 1, 0)^t$, $\vec{v}_3 = (1, 1, 1, 1)^t \in \mathbb{R}^4$, and let $L = \langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle$ be the linear subspace of $\mathbb{R}^4$ spanned by these vectors.
- (a) Find an orthonormal basis of L using the Gram-Schmidt procedure.
- (b) Find the orthogonal projection $P_L(\vec{y})$ of $\vec{y} = (1, 2, 3, 4)^t$.
- [2] 5. Verify that $\vec{v}_1 = (0, -1, 0, 0)$, $\vec{v}_2 = (1, 0, 0, 0)$, $\vec{v}_3 = (0, 0, 0, -1)$ and $\vec{v}_4 = (0, 0, 1, 0)$ form an orthonormal basis of $\mathbb{R}^4$, and use this fact to find the inverse of the matrix $A = (\vec{v}_1 | \cdots | \vec{v}_4)$.
- [5] 6. By following the steps below, use MAPLE (and the Linear Algebra package) to find the unique *cubic* polynomial $f(x)$ which (in the least square sense) best fits the data points

$$\{(-2, 2), (-1, 3), (0, 7), (1, 5), (2, 0)\}.$$

(a) Use the MAPLE commands `VandermondeMatrix` and `SubMatrix` to construct the 5×4 matrix A (of the least square formula) for the above data points. In addition, use the command `Vector` to write down the vector $\vec{y}$. [Reminder: to call MAPLE's Linear Algebra package, use first the command `with(LinearAlgebra)`.]

(b) Calculate the matrices $B = (A^t A)^{-1}$, $C = A^t \vec{y}$ and $\vec{w} = BC$. [Hint: use the MAPLE commands `Transpose` and `Multiply`.]

(c) Compare your result of part (b) with that obtained by using the MAPLE command `LeastSquares(A, y)`.

(d) From (b) (or (c)), write down the best fitting cubic f .

Suggestion: Before starting this MAPLE question, read the MAPLE handout and/or work through MAPLE Lab #2 to see how to use the *LinearAlgebra* package.