Question 1 (19 pts)

A Polynomial: ~~4 f~~ leading coefficient: 4

Roots: (2, 0) (-4, 0) and (6, 0)

Degree: 3

The polynomial is: $4(x-2)(x+4)(x-6)$

$$p(x) = 4x^3 - 16x^2 - 80x + 192.$$

(B) Difference of cubes at $x=2$.

$$\Rightarrow x^3 - (2)^3 = x^3 - 8, = (x-2)(x^2+2x+4)$$

$q(x) = x^3 - 8$

(C) $r(x) = \dfrac{p(x)}{q(x)} = \dfrac{4x^3 - 16x^2 - 80x + 192}{x^3-8}$

$$\cancel{= 4(6)^3 - 16(6)}$$

(i) $\lim\limits_{x\to 6} r(x) = \lim\limits_{x\to 6} \dfrac{4x^3 - 16x^2 - 80x + 192}{x^3-8}$

$$= \frac{4(6)^3 - 16(6)^2 - 80(6) + 192}{6^3 - 8}$$

$$= \frac{\cancel{256}\ 864 - 576 - 480 + 192}{216 - 8} = 0$$

$$\underline{\underline{= 0}}$$

(ii) $\lim\limits_{x\to 2} r(x) = \lim\limits_{x\to 2} \dfrac{4x^3 - 16x^2 - 80x + 192}{x^3 - 8}$

$$= \lim_{x\to 2} \frac{4(x-2)(x+4)(x-6)}{(x-2)(x^2+2x+4)}$$

$$= \lim_{x \to 2} \frac{4(x+4)(x-6)}{x^2+2x+4} = \frac{24(-4)}{4+4+4} = \frac{-96}{12}$$

$$= -8$$

(iii) $$\lim_{x \to \pm\infty} r(x) = \lim_{x \to \pm\infty} \frac{4x^3 - 16x^2 - 80x + 192}{x^3 - 8}$$

$$= \lim_{x \to \pm\infty} \frac{4 - \frac{16}{x} - \frac{80}{x^2} + \frac{192}{x^3}}{1 - \frac{8}{x^3}}$$

$$= \frac{4}{1} = 4$$

(D) Calculate

(i) $p'(1)$

$$P'(x) = 12x^2 - 32x - 80$$

$$p'(1) = 12 - 32 - 80 = -100$$

(ii) $q'(1)$

$$q'(x) = \cancel{24x^2}\ 3x^2$$

$$q'(1) = 3$$

(iii) $r'(1)$

$$r'(x) = \frac{(x^3-8)\cdot(12x^2-32x-80) - \overset{(4x^3-16x^2-80x+192)}{\cancel{(12x^2-32x-80)}}\, 3x^2}{(x^3-8)^2}$$

$$r'(1) = \frac{(1-8)\cdot(-100) - (4-16-80+192)(3)}{(1-8)^2}$$

$$= \frac{(-7)(-100) - (100)\cdot(3)}{(-7)^2}$$

$$= \frac{700 - 300}{49} = \frac{400}{49}.$$

Question 2 (13 points)

(A) $(-3,0)$; $(0,2)$

$$g = \frac{\Delta y}{\Delta x} = \frac{0-2}{-3-0} = \frac{-2}{-3} = \frac{2}{3}.$$

$$\frac{y-2}{x-0} = \frac{2}{3} \Leftrightarrow 3y-6 = 2x.$$

$$\Leftrightarrow 3y = 2x+6.$$

$$y = \frac{2}{3}x+2.$$

$$\therefore g(x) = \frac{2}{3}x+2.$$

(B) $h(x) = \dfrac{ax+b}{x+c}$.

$$\lim_{x\to\pm\infty} \frac{ax+b}{x+c} = 2 \Leftrightarrow \lim_{x\to\pm\infty} \frac{a+\frac{b}{x}}{1+\frac{c}{x}} = 2. \qquad \underline{\underline{a=2}}$$

$$\left.\begin{array}{l} \lim_{x\to 4^+} h(x) = \infty \\ \lim_{x\to 4} h(x) = -\infty \end{array}\right\} \text{vertical asymptote at } x = 4$$

$$\therefore x+c \Leftrightarrow x-4$$

$$\therefore c = -4$$

Now $h(x) = \dfrac{2x+b}{x-4}$

but $h(1) = 4 \Leftrightarrow \dfrac{2+b}{1-4} = 4.$

$$\Leftrightarrow 2+b = -12. \qquad b = -14$$

$\therefore$ $a = 2$, $b = -14$ and $c = -4$

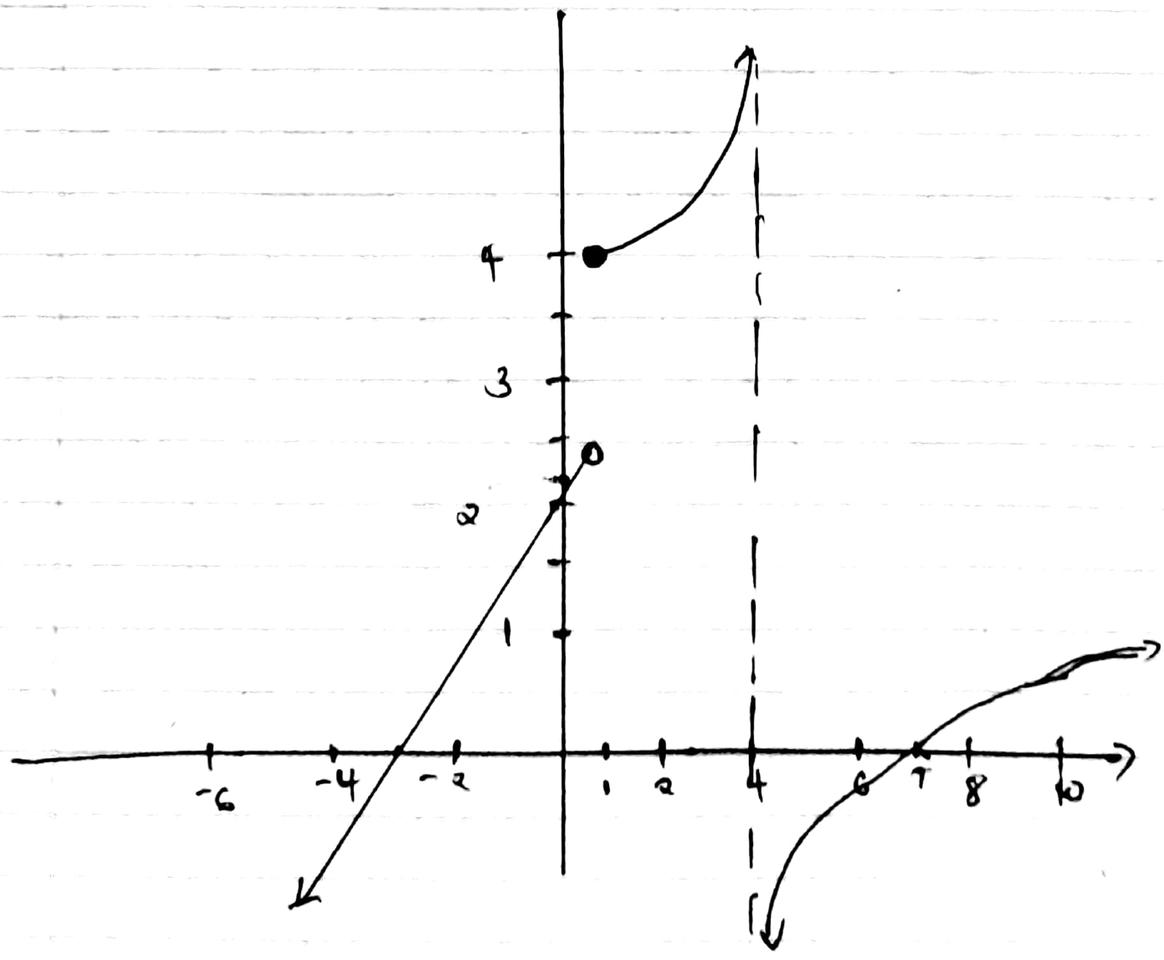
$$\Leftrightarrow h(x) = \frac{2x-14}{x-4}$$

(c) $f(x) = \begin{cases} g(x) & \text{if } x < 1 \\ h(x) & \text{if } x \geq 1 \end{cases}$

The function becomes:

$$f(x) = \begin{cases} \frac{2}{3}x + 2 & \text{if } x < 1 \\ \frac{2x-14}{x-4} & \text{if } x \geq 1 \end{cases}$$

Sketch.



$f(x)$ is discontinous at:

$x = 1$

$x = 4$ because $\lim\limits_{x \to 4^+} f(x) \neq \lim\limits_{x \to 4^-} f(x)$

Question 3 (13 points)

(A) $y = x^2$

Reflect around the x-axis: $y = -x^2$

shifting up by 10 units: $\boxed{y = -x^2 + 10}$

$\therefore \boxed{g(x) = -x^2 + 10}$

(B) Cub root of x : $\Rightarrow \sqrt[3]{x}$ or $x^{1/3}$

$$\therefore \boxed{f(x) = x^{1/3}}$$

(C) Determine:

(i) $h(x) = f \circ g(x)$

$$= f(g(x)) = f(10 - x^2)$$

$$= (10 - x^2)^{1/3}$$

(ii) $r(x) = \dfrac{f(x)}{gx} = \dfrac{x^{1/3}}{10 - x^2}$

(iii) $P(x) = (f(x))^4 \cdot (g(x))^5$

$$= (x^{1/3})^4 \cdot (10 - x^2)^5$$

$$= x^{4/3} \cdot (10 - x^2)^5$$

D Compute:

(i) $f'(x)$ and $g'(x)$

$$f'(x) = f'(x^{1/3}) = \tfrac{1}{3} x^{-2/3} \text{ OR } \frac{1}{3x^{2/3}}$$

$$g'(x) = g'(10 - x^2)$$

$$= -2x$$

(ii) $h'(x)$

$h(x) = (10 - x^2)^{1/3}$. Using chain rule,

$h'(x) = \frac{1}{3}(10 - x^2)^{-2/3} \cdot (-2x)$

$= -\frac{2}{3}x\,(10 - x^2)^{-2/3}$

(iii) $r'(x)$ and $p'(x)$

$r(x) = \dfrac{x^{1/3}}{10 - x^2}$ Using quotient rule,

$r'(x) = \dfrac{(10 - x^2)\cdot \frac{1}{3}x^{-2/3} - x^{1/3}(-2x)}{(10 - x^2)^2}$

$= \dfrac{(10 - x^2)\cdot \frac{1}{3}x^{-2/3} + 2\cdot x^{2/3}}{(10 - x^2)^2}$

~~p'x~~ $p(x) = x^{4/3}\cdot(10 - x^2)^5$

Using product rule,

$p'(x) = \{x^{4/3}\cdot 5(10 - x^2)^4 \cdot -2x\} + \{(10 - x^2)^5 \cdot \frac{4}{3}x^{1/3}\}$

$= \frac{4}{3}x^{1/3}(10 - x^2)^5 - 10x^{7/3}(10 - x^2)^5.$

Question 4

(A) $F(t) = pe^{0.02t}$.

(B) when the property doubles, $f(t) = 2p$.

$\therefore 2p = pe^{0.02t}$ Divide throug by p to get.

$2 = e^{0.02t}$. Introduce ln both sides.

$\ln 2 = \ln[e^{(0.02t)}]$

$\ln 2 = 0.02t$.

$t = \frac{\ln 2}{0.02} = 34.7$ years.

(C) $F(t) = 250{,}000\,e^{0.02t}$.

$F'(t) = 0.02 \times 250{,}000\,e^{0.02t}$.

$f'(t) = 5{,}000\,e^{0.02t}$.

$f'(5) = 5000\,e^{0.1}$.

$= 5525.85$.

Question 5

(A) $e^{-x^2} = fx$ $\Leftrightarrow$ $\cancel{e^{-x^2}}$ $fx = \frac{1}{e^{x^2}}$.

$\Leftrightarrow$ $\cancel{\frac{x^2}{e}}$ Domain: $(-\infty, \infty)$

(B) No vertical ~~asy~~ asymptote.

(C) ~~Hor~~ Horizontal asymptote: $y = 0$.

$$\lim_{x \to \infty} = \frac{1}{e^{x^2}} = \frac{1}{\infty} = 0.$$

(D) Critical points of $f'(x)$.

$$f'(x) = -2x \cdot e^{-x^2}.$$

$$f''(x) = -2x \cdot -2x e^{-x^2} + -2e^{-x^2}.$$

$$= 4x^2 e^{-x^2} - 2e^{-x^2} = 0.$$

$$e^{-x^2}(4x^2 - 2) = 0.$$

$$4x^2 - 2 = 0.$$

$$x^2 = 2/4 \quad \text{and } x = \pm \frac{\sqrt{2}}{2}.$$

$\therefore$ $x = -\frac{\sqrt{2}}{2}$ and $x = \frac{\sqrt{2}}{2}$

E $f'(x) = -2xe^{-x^2}$

If $f'(x) > 0$ then $f(x)$ is Increasing.

$-2xe^{-x^2} > 0$ $\Leftrightarrow$ $xe^{-x^2} < 0$ $\Leftrightarrow$ $x < 0$.

for $f'(x) < 0$; $-2xe^{-x^2} < 0$ $\Leftrightarrow$ $x > 0$.

Combine the domain:

$-\infty < x < 0, \quad 0 < x < \infty$

	$-\infty < x < 0$	$0 < x < \infty$
Sign	$f'(x) > 0$	$f'(x) < 0$
Behaviour	Increasing	Decreasing

$\therefore$ Intervals of Increasing: $-\infty < x < 0$ and Intervals of decreasing is: $0 < x < \infty$.

(F) $f(x) = e^{-x^2}$ $\quad f'(x) = -2x e^{-x^2}$

$f''(x) = 4x^2 e^{-x^2} - 2e^{-x^2}$.

~~Critical points~~ $f''(x)$ is undefined when:

$$4e^{-x^2}(4x^2 - 2) = 0$$

$$x = -\frac{\sqrt{2}}{2} \text{ and } x = \frac{\sqrt{2}}{2}.$$

Domain: $(-\infty, \infty)$

Hence the intervals are: $\left(-\infty, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}, \infty\right)$

Check the sign of $f''(x)$ at each interval:

Sign $\cancel{\left(-\infty, \frac{-2}{\sqrt{2}}\right)}$

~~behaviour~~

behaviour

	$(-\infty, -\frac{\sqrt{2}}{2})$	$x = -\frac{\sqrt{2}}{2}$	$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$x = \frac{\sqrt{2}}{2}$	$(\frac{\sqrt{2}}{2}, \infty)$
sign	+	0	-	0	+
behavior	Concave up	Inflection	concave downward	Inflection	Concave Upward.

Intervals of concavity:

Concave upward: $-\infty < x < -\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{2} < x < \infty$

Concave downward: $\left(-\frac{\sqrt{2}}{2}, \frac{2}{\sqrt{2}}\right)$

Points of Inflection: $\left(-\frac{\sqrt{2}}{2}, 0.61\right)$

$\left(\frac{\sqrt{2}}{2}, 0.6\right)$

G. X-intercept. $(y=0)$

$e^{-x^2} = 0.$ $\quad \ln(0) = -x^2$

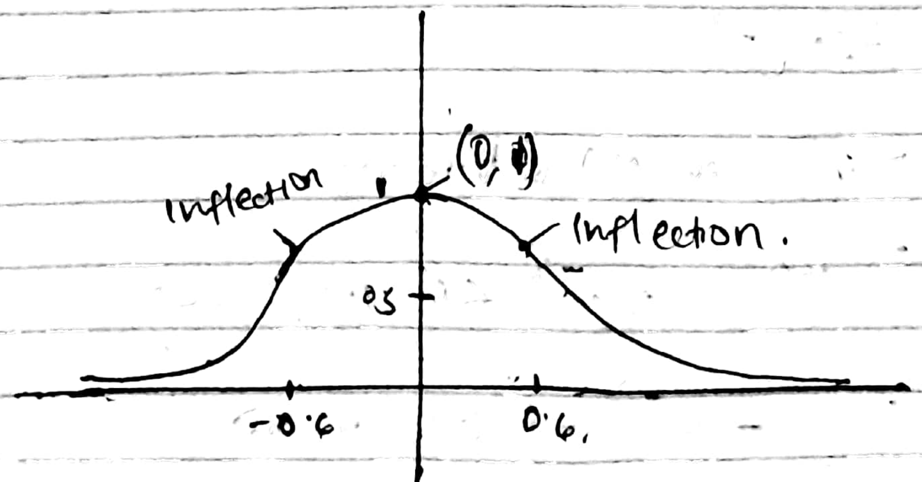
but $\ln(0)$ DNE

$\therefore$ NO x-intercept.

Y-intercept: $x=0.$

$\Rightarrow e^0 = 1 \quad \Rightarrow (0, 1)$

H.



Question 6

$f(x) = 3\sqrt{x}$ $\quad g(x) = \frac{1}{3}(x)$

First, determine the points of intersection.

$3\sqrt{x} = \frac{1}{3}x$

$9\sqrt{x} = x$. square both sides

$81x = x^2$

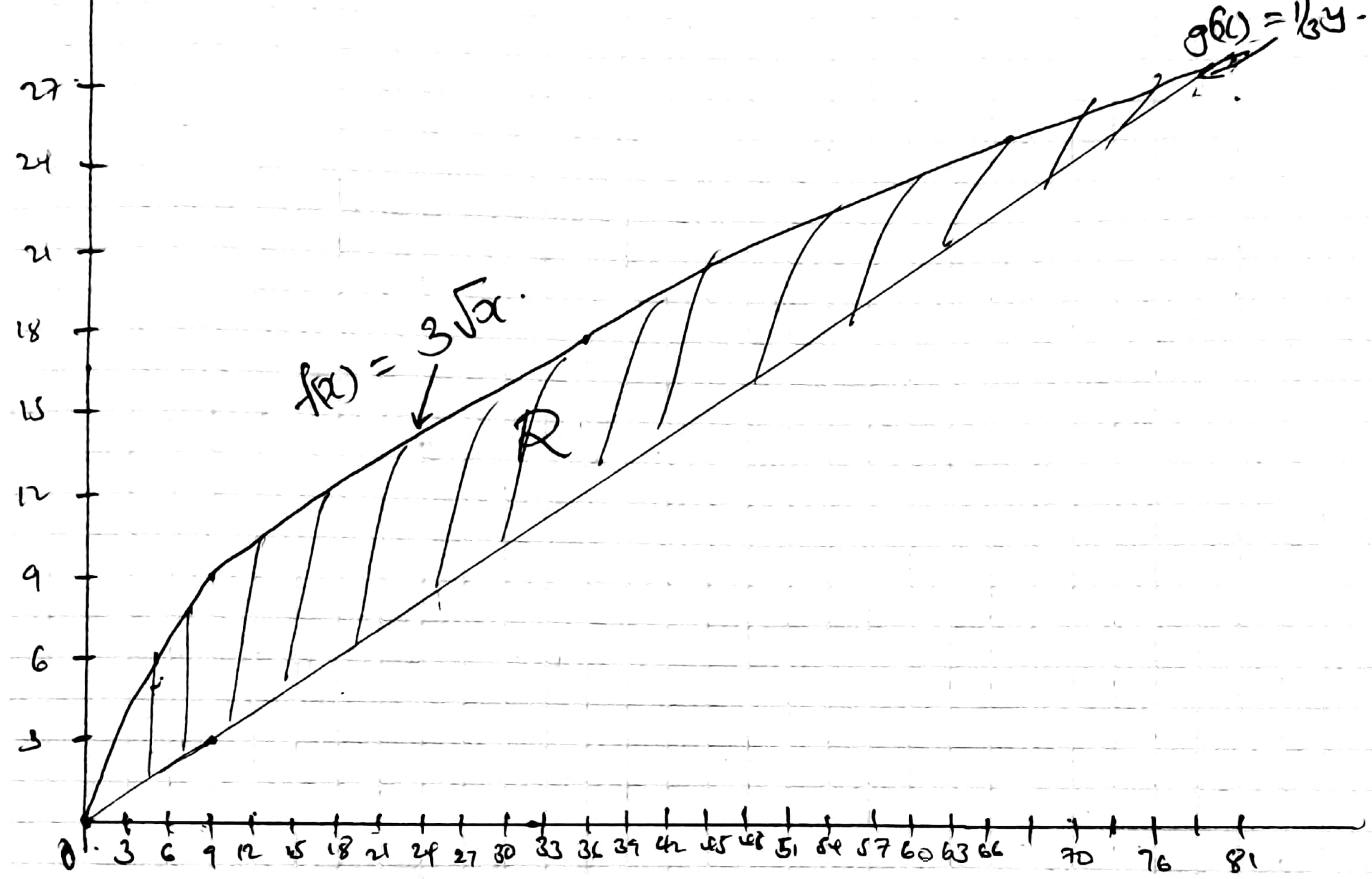
$(x^2 - 81x) = 0$

$x(x - 81) = 0$ (2) $x = 0,\ x = 81$

Plotting the region.

x	fx
4	6
9	9
16	12
25	15
36	18
49	21
64	24
81	27

x	$g(x)$
3	1
6	2
9	3
12	4



$f(x) = 3\sqrt{x}$
$g(x) = \frac{1}{3}y$
R

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

$$= \int_0^{81} \left|3\sqrt{x} - \frac{1}{3}x\right| dx.$$

$$= \int_0^{81} \left| 3x^{1/2} - \frac{1}{3}x \right| dx.$$

$$= \left| \left[2x^{3/2} - \frac{1}{6}x^2 \right]_0^{81} \right|$$

$$= \left\{ \left(2(81)^{3/2} - \frac{1}{6}(81)^2 \right) - 0 \right\} = \cancel{-\frac{729}{2}}$$

$$= \left| 1458 - \frac{2187}{2} \right| = \left| -\frac{729}{2} \right|$$

$$= \frac{729}{2}$$