

Project 7 (001): The SIR Epidemic Model

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This project explores the how we can model the outbreak of an epidemic

Assume we have a population with a fixed size N . There is an infection spreading through the population and we're going to divide the population into three groups: Susceptible, Infected and Recovered. At a given moment in time t ,

$$S(t) = \text{Number who are have not had the infection but are susceptible}$$
$$I(t) = \text{Number infected}$$
$$R(t) = \text{Number who have had the infection but have recovered}$$

We will assume that once you have had the infection you cannot get it again. At any given moment in time $S(t) + I(t) + R(t) = N$. Notice this means that $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.

We make the following assumptions:

- Those who are susceptible can become infected. Those who are infect can recover. Hence the number of susceptibles decreases and the number of recovered increases.
- The rate of change of S (with respect time) is directly proportional to both I and S .
- The rate of change of R (with respect to time) is directly proportional to I

These three assumptions (along with $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$.) imply that this system is governed by the following differential equations:

$$\begin{aligned}\frac{dS}{dt} &= -\beta SI \\ \frac{dI}{dt} &= \beta SI - \nu I \\ \frac{dR}{dt} &= \nu I\end{aligned}$$

where $\beta > 0$ is called the transmission rate and $\nu > 0$ is called the recovery rate.

Unfortunately we will not be able to find explicit solutions to these equations in general. However, we will be able prove various important properties of solutions to such a system.

1. a) The monotone convergence theorem implies that if a function f is monotone (increasing or decreasing) and bounded on $[0, \infty)$ then $\lim_{t \rightarrow \infty} f(t)$ exists.

Use this to show that $\lim_{t \rightarrow \infty} S(t)$, $\lim_{t \rightarrow \infty} I(t)$ and $\lim_{t \rightarrow \infty} R(t)$ all exist. Denote these limits by S_∞ , I_∞ and R_∞ . This means the situations stabilizes over time. Hint: Consider S and R first and remember the total population size is N .

Solution:

- b) It is a fact that if $\lim_{t \rightarrow \infty} f'(t) > 0$ then $\lim_{t \rightarrow \infty} f(t) = \infty$.

Use this to prove that $I_\infty = 0$. This means the epidemic will always end. Nice!

Hint: Consider $f(t) = R(t)$

2. a) Recall that S is a decreasing function. Use this to prove that if $\beta S(0) - \nu < 0$ then $\frac{dI}{dt} < 0$ for all t . This shows that under this condition, $I(t)$ is a decreasing function and there will not be an epidemic. Hint: First factor out I from the right-hand-side of the differential equation involving $\frac{dI}{dt}$.

Solution:

- b) Following from the previous question, show that if $\beta S(0) - \nu > 0$, then $I(t)$ will rise to a maximum before decreasing to zero. Under these circumstances there will be an epidemic. The larger this value the worse it will be.

Solution:

3. a) Using the chain rule we know $\frac{(\frac{dI}{dt})}{(\frac{dS}{dt})} = \frac{dI}{dS}$. Use this to establish a differential equation involving $\frac{dI}{dS}$. Solve this differential equation to prove

$$I(t) = -S(t) + \nu/\beta \ln(S(t)) + I(0) + S(0) - \nu/\beta \ln(S(0)).$$

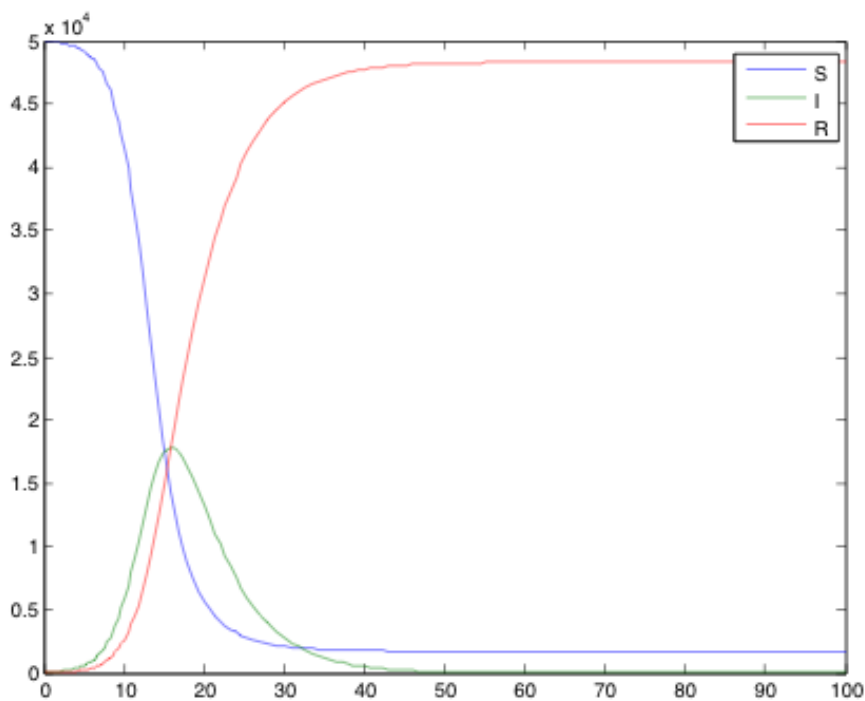
Solution:

- b) Assume that $\beta S(0) - \nu > 0$. This means that there is some time $t_0 > 0$, when I is at a maximum value denoted by I_{max} . This must happen when $\frac{dI}{dt} = 0$. Using our system of differential equations, prove that $\beta S(t_0) - \nu = 0$. Using this and the previous question, prove that

$$I_{max} = I(0) + S(0) - \nu/\beta \ln(S(0)) - \nu/\beta + \nu/\beta \ln(\nu/\beta).$$

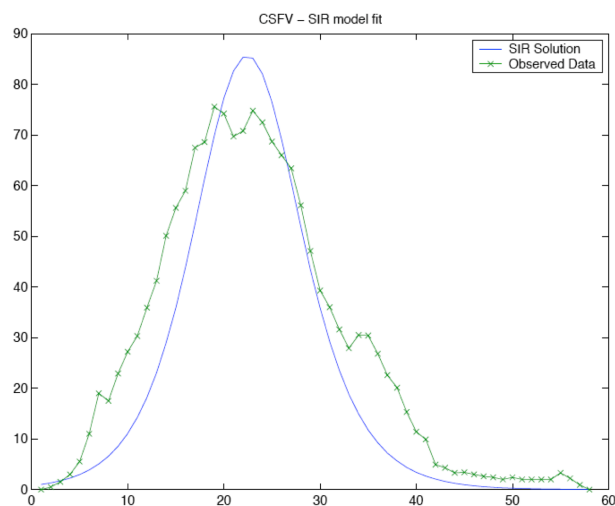
Solution:

To illustrate this scenario we can simulate a highly virulent flue epidemic in a town of 50000.



In this simulation $\beta = 0.7/50000$, $\nu = 1/5$, $S(0) = 49955$, $I(0) = 5$ and $R(0) = 0$.

Though fairly simple, the SIR model has been found to simulate epidemics with impressive accuracy. Here's an example of the SIR prediction versus the observed data in a swine fever outbreak in the Netherlands in 1997-1998.



The SIR model was created in 1927. Nowadays there are much more sophisticated models taking into account more subtle properties of how infection diseases spread. If you're interested I'd recommend looking at the Wikipedia page on *Mathematical modelling of infection disease*.