

INTEGRATION BY SUBSTITUTION

12.47

a) $\int 252x^2(7x^3-12)^2 dx$

Substitute x^3 by u

$$= 252 \int \frac{(7u-12)^2}{3} du; \quad du = 3x^2 dx, \quad dx = \frac{1}{3x^2} du$$

which gives

$$\int x^2 (7u-12)^2 \frac{1}{3x^2} du = \frac{(7u-12)^2}{3} du$$

$$= 252 \cdot \frac{1}{3} \int (7u-12)^2 du$$

Expanding $(7u-12)^2$ to get $(7u)^2 - 2(7u)(12) + 12^2 = 49u^2 - 168u + 144$

$$= 252 \cdot \frac{1}{3} \int 49u^2 - 168u + 144 du$$

$$= 252 \cdot \frac{1}{3} \left(\int 49u^2 du - \int 168u du + \int 144 du \right)$$

$$= 252 \cdot \frac{1}{3} \left[\frac{49u^3}{3} - 84u^2 + 144u \right]$$

$$= 252 \cdot \frac{1}{3} \left(\frac{49u^3}{3} - 84u^2 + 144u \right)$$

But $u = x^3$

$$= 252 \cdot \frac{1}{3} \left(\frac{49(x^3)^3}{3} - 84(x^3)^2 + 144x^3 \right)$$

$$= 84 \left(\frac{49x^9}{3} - 84x^6 + 144x^3 \right) + C$$

b) $\int 90x^3(3x^4+19)^4 dx$

Substitute x^4 by u

$$= 90 \int \frac{u^{1/3} (3u^{4/3} + 19)^4}{3} du$$

$$= 90 \cdot \frac{1}{3} \int u^{1/3} (3u^{4/3} + 19)^4 du$$

Expanding $(3u^{4/3} + 19)^4$ to give $81u^{16/3} + 2052u^{14/3} + 1949u^{12/3} + 82308u^{10/3} + 13032u^{8/3} + 13032u^{4/3} + 13032u^{2/3} + 13032$

$$= u^{1/3} (2052 u^4 + 81 u^{16/3} + 19494 u^{8/3} + 130321)$$

$$= 81 u^{16/3} u^{1/3} + 2052 u^4 u^{1/3} + 19494 u^{8/3} u^{1/3} + 82308 u^{5/3} u^{1/3} + 130321 u^{1/3}$$

$$= 81 u^{17/3} + 2052 u^{13/3} + 19494 u^3 + 82308 u^{5/3} + 130321 u^{1/3}$$

$$= 90 \cdot \frac{1}{3} \int (81 u^{17/3} + 2052 u^{13/3} + 19494 u^3 + 82308 u^{5/3} + 130321 u^{1/3}) du$$

$$= 90 \cdot \frac{1}{3} \left(\frac{243}{20} u^{20/3} + \frac{1539}{4} u^{16/3} + \frac{9747}{2} u^4 + \frac{61731}{2} u^{8/3} + \frac{390963}{4} u^{4/3} \right)$$

But $u = x^3$

$$= 90 \cdot \frac{1}{3} \left(\frac{243}{20} (x^3)^{20/3} + \frac{1539}{4} (x^3)^{16/3} + \frac{9747}{2} (x^3)^4 + \frac{61731}{2} (x^3)^{8/3} + \frac{390963}{4} (x^3)^{4/3} \right)$$

$$= 30 \left(\frac{9747 x^{12}}{2} + \frac{243}{20} x^{20} + \frac{1539}{4} x^{16} + \frac{61731}{2} x^8 + \frac{390963}{4} x^4 \right) + C$$

c) $\int 672x(8x^2-9)^5 dx$

$$= 672 \int x(8x^2-9)^5 dx$$

Apply u-substitution, $u = 8x^2 - 9$

$$= 672 \int \frac{u^5}{16} du$$

$$= 672 \cdot \frac{1}{16} \int u^5 du$$

$$= 672 \cdot \frac{1}{16} \cdot \frac{(8x^2-9)^6}{6}$$

$$= 7 (8x^2-9)^6 + C$$

d) $\int \frac{120x^3}{(5x^4-18)^4} dx = 120 \int \frac{x^3}{(5x^4-18)^4} dx$

Substitute $5x^4-18$ by u

$$= 120 \int \frac{1}{20u^4} du$$

$$= \frac{120}{20} \int \frac{1}{u^4} du = \frac{120}{20} \int \frac{(5x-18)^{-4+1}}{-4+1}$$

$$= -\frac{2}{(5x^4-18)^3} + C$$

$$\textcircled{2} \int \frac{-576x^2}{(8x+13)^5} dx = -576 \int \frac{x^2}{(8x+13)^5} dx$$

Substitute $u = 8x+13$, $\frac{du}{dx} = 8$, $dx = \frac{1}{8} du$

$$x^2 = \left(\frac{u-13}{8}\right)^2 du$$

$$\int \left(\frac{u-13}{8}\right)^2 du = \int \left(\frac{1}{8^3} - \frac{26}{8^2}u + \frac{169}{8}u^2\right) du$$

$$= \int \frac{1}{8^3} du - \frac{26}{8^2} \int u du + \frac{169}{8} \int \frac{1}{u^3} du$$

$$= -\frac{1}{1024u^2} + \frac{26}{768u^3} - \frac{169}{2048u^4}$$

$$= \frac{1}{1024u^2} + \frac{13}{768u^3} - \frac{169}{2048u^4}$$

But $u = 8x+13$

$$= -\frac{1}{1024(8x+13)^2} + \frac{13}{768(8x+13)^3} - \frac{169}{2048(8x+13)^4}$$

$$= \frac{9}{16(8x+13)^2} - \frac{39}{4(8x+13)^3} + \frac{1521}{32(8x+13)^4}$$

$$= \frac{3(384x^2 + 416x + 169)}{32(8x+13)^4} + C$$

$$= -576 \left(-\frac{1}{1024(8x+13)^2} + \frac{13}{768(8x+13)^3} - \frac{169}{2048(8x+13)^4} \right) + C$$

$$\begin{aligned}
 8.9) \quad \int \frac{30x^4}{6x^5-21} dx &= 30 \int \frac{x^4}{6x^5-21} dx \\
 \text{Use } u \text{ substitution: } u &= 6x^5-21 \\
 &= 30 \int \frac{x^4}{30u} du \\
 &= 30 \cdot \frac{1}{30} \int \frac{x^4}{u} du \\
 &= 30 \cdot \frac{1}{30} \ln |u| \\
 &= 30 \cdot \frac{1}{30} \ln |6x^5-21| \\
 &= \ln |6x^5-21| + C
 \end{aligned}$$

$$\begin{aligned}
 9) \quad \int \frac{18x^3}{\sqrt{9x^4-5}} dx &= 18 \int \frac{x^3}{\sqrt{9x^4-5}} dx \\
 \text{Let } u &= 9x^4-5 \\
 &= 18 \int \frac{1}{36\sqrt{u}} du \\
 &= 18 \cdot \frac{1}{36} \int \frac{1}{u^{1/2}} du \\
 &= 18 \cdot \frac{1}{36} \cdot \frac{(9x^4-5)^{1/2}}{1/2} \\
 &= 18 \times 2 \cdot (9x^4-5)^{1/2} \\
 &= \frac{1 \times 18 \times 2}{36} (9x^4-5)^{1/2} = (9x^4-5)^{1/2} \\
 &= \sqrt{9x^4-5} + C
 \end{aligned}$$

$$\begin{aligned}
 10) \quad \int 99x^2 \sqrt{22x^3+19} dx &= 99 \int x^2 \sqrt{22x^3+19} dx \\
 \text{Let } u &= \sqrt{22x^3+19} \\
 &= 99 \int \frac{u^2}{33} du \\
 &= 99 \cdot \frac{1}{33} \int u^2 du \\
 &= 99 \cdot \frac{1}{33} \left[\frac{u^3}{3} \right] du \\
 &= 99 \cdot \frac{1}{33} \left(\frac{(22x^3+19)^{3/2}}{3} \right) = \frac{(22x^3+19)^{3/2}}{3} + C
 \end{aligned}$$

$$i) \int 96x^2 e^{4x^3} dx = 96 \int x^2 e^{4x^3} dx$$

$$\text{let } u = 4x^3$$

$$= 96 \cdot \int \frac{e^u}{12} du$$

$$= 96 \cdot \frac{1}{12} e^u \quad \because \int e^u du = e^u$$

$$= 96 \cdot \frac{1}{12} e^{4x^3}$$

$$= \underline{\underline{8e^{4x^3} + C}}$$

$$j) \int 120x^3 e^{-6x^4} dx = 120 \int x^3 e^{-6x^4} dx$$

$$\text{let } u = -6x^4$$

$$= 120 \left(-\frac{1}{24} \int e^u du \right), \quad \because \int e^u du = e^u$$

$$= 120 \left(-\frac{1}{24} e^{-6x^4} \right)$$

$$= \frac{-1 \times 120}{24} e^{-6x^4}$$

$$= \frac{-1 \times 120}{24} = -5$$

$$= \underline{\underline{5e^{-6x^4} + C}}$$

$$a) \int 20x(x+9)^4 dx$$

$$= 20 \int 2(x+9)^4 dx = 20 \int (x^5 + 36x^4 + 486x^3 + 2916x^2 + 6561x) dx$$

$$= 20 \int (x^5 + 36x^4 + 486x^3 + 2916x^2 + 6561x) dx$$

$$= 20 \left(\frac{x^6}{6} + \frac{36x^5}{5} + \frac{486x^4}{4} + \frac{2916x^3}{3} + \frac{6561x^2}{2} \right) + C$$

$$= 20 \left(\frac{x^6}{6} + \frac{36x^5}{5} + \frac{243x^4}{2} + 972x^3 + \frac{6561x^2}{2} \right) + C$$

$$= \underline{\underline{20 \left(\frac{x^6}{6} + \frac{36x^5}{5} + \frac{243x^4}{2} + 972x^3 + \frac{6561x^2}{2} \right) + C}}$$

$$b) \int 42x(x-3)^5 dx = 42 \int x(x-3)^5 dx$$

$$\text{Expand } (x-3)^5 = x^6 - 15x^5 + 90x^4 - 270x^3 + 405x^2 - 243x$$

$$= 42 \int (x^6 - 15x^5 + 90x^4 - 270x^3 + 405x^2 - 243x) dx$$

$$= 42 \left(\frac{x^7}{7} - \frac{15x^6}{6} + \frac{90x^5}{5} - \frac{270x^4}{4} + \frac{405x^3}{3} - \frac{243x^2}{2} \right) + C$$

$$c) \int \frac{36x}{(x+8)^5} dx = 36 \int \frac{x}{(x+8)^5} dx$$

$$\text{let } u = x+8$$

$$= 36 \int \frac{u-8}{u^5} du$$

$$\text{Expand } \frac{u-8}{u^5} \text{ to get } \frac{1}{u^4} - \frac{8}{u^5}$$

$$= 36 \int \frac{1}{u^4} - \frac{8}{u^5} du$$

$$= 36 \left(\int \frac{1}{u^4} du - \int \frac{8}{u^5} du \right)$$

$$= 36 \left(-\frac{1}{3u^3} + \frac{2}{u^4} \right)$$

Substitute the value of u back

$$= 36 \left(-\frac{1}{3(x+8)^3} + \frac{2}{(x+8)^4} \right) + C$$

$$d) \int 45x\sqrt{x+2} dx = 45 \int x\sqrt{x+2} dx$$

$$\text{let } u = x+2$$

$$= 45 \int (u-2)\sqrt{u} du$$

$$\text{Expand } (u-2)\sqrt{u} = 45 \int u^{3/2} - 2u^{1/2} du$$

$$= 45 \left(\frac{2}{5} u^{5/2} - 14 u^{3/2} \right)$$

$$= 45 \left(\frac{2}{5} (x+2)^{5/2} - 14 (x+2)^{3/2} \right) + C$$

$$e) \int 7x e^{x-3} dx = 7 \int x e^{x-3} dx$$

$$u = x, v = e^{x-3}$$

$$= 7 \left(e^{x-3} x - \int e^{x-3} dx \right)$$

$$= \int e^{x-3} dx = e^{x-3}$$

$$\int e^{x-3} dx = \int e^{-3} e^x dx = e^{-3} \int e^x dx = \frac{1}{e^3} \int e^x dx$$

$$= \frac{1}{e^3} e^x = e^{x-3}$$

$$= 7 \left(e^{x-3} x - e^{x-3} \right) + C$$

$$f) \int 32x^2 e^{4x} dx = 32 \int x^2 e^{4x} dx$$

let $u = x^2$ and $v' = e^{4x}$

$$= 32 \left(\frac{1}{4} e^{4x} x^2 - \int \frac{1}{2} e^{4x} x dx \right)$$

$$\int \frac{1}{2} e^{4x} x dx = \frac{1}{32} (4e^{4x} x - e^{4x})$$

$$= 32 \left(\frac{1}{4} e^{4x} x^2 - \frac{1}{32} (4e^{4x} x - e^{4x}) \right)$$

$$= 8e^{4x} x^2 - 4e^{4x} x + e^{4x} + C$$

$$12.49 \quad MC = x^2 - 11x + 385$$

$$FC = 450$$

Tc function = ?

$$TVC = \int MC = \int x^2 - 11x + 385 dx$$

$$= \frac{x^3}{3} - 5.5x^2 + 385x + C$$

$$TC = TFC + TVC$$

$$= 450 + 385x - 5.5x^2 + \frac{x^3}{3}$$

$$12.50) \quad TC = \int MC = x^3 - 2x^2 + 525x \quad \text{where } MC = 3x^2 - 4x + 525$$

$$= 64 - 32 +$$

$$TC_4 = (4)^3 - 2(4)^2 + 525(4)$$

$$= 64 - 32 + 2100$$

$$= 2068$$

$$TC_7 = 7^3 - 2(7)^2 + 525(7)$$

$$= 3920$$

$$\Delta TC = TC_7 - TC_4 = 3920 - 2068 = 1852$$

$$12.51 \quad TR = \int MR = -0.08x^3 - 0.75x^2 + 660x$$

$$12.52 \quad MR = 104 - 6x$$

$$TR = 104x - 3x^2$$

$$TR_{x=10} = 104(10) - 3(10)^2 = 740$$

$$TR_{x=12} = 104(12) - 3(12)^2 = 816$$

$$\text{Increase in TR} = 816 - 740 = \underline{76}$$

12.53 Given rate, $v'(t) = 250(t-6)$, $0 \leq t \leq 6$

a) Total amount that can depreciates in first 3 years.

$$v'(t) = 250(t-6)$$

$$\frac{dv}{dt} = 250(t-6) \Rightarrow dv = 250(t-6) dt$$

Integrating both sides, $t=0$ to $t=3$

$$\int_0^3 dv = \int_0^3 250(t-6) dt$$

$$\int ax^n dx = \frac{a^{n+1}}{n+1}$$

$$v = 25 \left[\frac{t^2}{2} - 6t \right]_0^3$$

$$= 25 \left[\frac{9}{2} - 18 \right] = 25 \left[-\frac{27}{2} \right] = -337.5$$

$\therefore$ Amount will depreciate by 337.5

b) For the last 3 years = for first 6 years - for first 3 years.

$$= \int_0^6 250(t-6) dt - \int_0^3 250(t-6) dt$$

$$= 250 \left[\frac{t^2}{2} - 6t \right]_0^6 - (-337.5)$$

$$= 250 [10 - 36] + 337.5$$

$$= -4162.5$$

Therefore, amount will decrease by 4,162.5

12.56. $x_0 = 6$, $p_0 = 83.4$ $p = 0.15x^2 + 8x + 30$

producer's surplus = $p \times q - \int_0^q S(q) dq$

$= 6 \times 83.4 - \int_0^6 (0.15x^2 + 8x + 30) dx$

$= 500.4 - 334.8 = 165.6$

165.6

12.54. $x_0 = 5$, $p_0 = 125$, $p = -6x^2 + 275$

Consumer surplus = $\int_0^q d(x) dx - p \times q$

$= \int_0^5 (-6x^2 + 275) dx - 5 \times 125$

$= 1125 - 625$

500