School of Mathematics, Computer Science and Engineering

Managing Risk and Uncertainty

May 2021

Answer **THREE OUT OF FOUR** questions
Division of marks: All questions carry equal marks.

BEGIN EACH QUESTION ON A FRESH PAGE

Number of answer books to be provided: ONE
Calculators permitted: Casio FX-83/85 MS/ES/GT+ ONLY
Examination duration: 120 minutes
Dictionaries permitted: None
Additional materials: None
Can question paper be removed from the examination room: No

Question 1

(a) There are n roads from A to B and n roads from B to C. Each of these $2n$ roads has probability p of being blocked, independently of the others.

(i) Discuss briefly union and intersection risk in the context of the situation described above. **[2 marks]**

(ii) Find the probability that there is an open road from A to C. **[4 marks]**

(b) Further to the assumptions of part (a), assume that there is also a direct road from A to C, which is independently blocked from the others with probability p. Find the new probability that there is an open road from A to C. **[4 marks]**

(c) Let $(X_1, X_2, \ldots, X_n)$ be independent and identically distributed random variables, with uniform distribution in the interval $(0, 1)$. Let $Y = \max(X_1, X_2, \ldots, X_n)$. Show that the probability density function of Y is $f_Y(y) = ny^{n-1}$, $0 < y < 1$. *Hint:* Work out first the cumulative distribution function of Y. **[5 marks]**

(d) Consider the revenue stream $(-10, 6, 8)$ corresponding to consecutive years. Find the Internal Rate of Return (IRR) of the stream. **[5 marks]**

Question 2

(a) Let L be a random variable representing loss with probability mass function given by the following table:

L	-10	10	20	100
Probability	0.5	0.4	0.06	0.04

Find $\mathrm{VaR}_{0.95}(L)$ and $\mathrm{CVaR}_{0.95}(L)$. **[5 marks]**

(b) Let L be a loss function with exponential density $f_L(x) = e^{-x}$ for $x \geq 0$ and $f_L(x) = 0$ for $x < 0$. Find $\mathrm{VaR}_\beta(L)$ and $\mathrm{CVaR}_\beta(L)$ at a level $\beta \in (0,1)$. *Hint:* $\int xe^{-x}dx = -(x+1)e^{-x}$. **[5 marks]**

(c) Let X and Y be two random variables representing investment losses. Show that if X dominates Y stochastically in first order, then $\mathrm{VaR}_\beta(X) \geq \mathrm{VaR}_\beta(Y)$ for every $\beta \in (0,1)$. *Note:* An explanation by means of a graph is acceptable. **[5 marks]**

(d) Let L be a loss function. The relation between $\mathrm{VaR}_\beta(L)$ and $\mathrm{CVaR}_\beta(L)$ at any confidence level $\beta \in (0,1)$ is given by:

$$\mathrm{CVaR}_\beta(L) = \frac{1}{1-\beta}\int_\beta^1 \mathrm{VaR}_u(L)du$$

(Note: you do not need to prove this formula). (i) Show that $\mathrm{CVaR}_\beta(L) \geq \mathrm{VaR}_\beta(L)$ for any $\beta \in (0,1)$. (ii) It is well known that $\mathrm{VaR}_\beta(L)$ satisfies the properties of monotonicity, translation invariance and positive homogeneity. (Note: Again, you are not asked to prove these properties). By considering the above relation show that $\mathrm{CVaR}_\beta(L)$ also satisfies these three properties. **[5 marks]**

Question 3

An oil company needs to decide whether or not to drill on a specific site. It will cost 85 million to drill. If the site turns out to be "dry", all 85 million will be lost. If the site turns out to be "wet", the profit will be 170 million; finally if the site turns out to be "very wet", the profit will be 660 million. The company's geologists estimate that for this particular site there is a 65% chance that the geological conditions are favourable for the presence of oil and a 35% chance that they are unfavourable. In the first case, the probabilities that the site is "dry", "wet" or "very wet" are 0.63, 0.24 and 0.13, respectively. These change to 0.87, 0.109 and 0.021, respectively, if the conditions are unfavourable.

(a) Draw the decision tree of the problem. **[5 marks]**

(b) Let X be a random variable representing the profit of the company if they decide to drill. Calculate the probability mass function and the expected value of X.

[5 marks]

(c) Suppose that after drilling the site turned out to be "wet". Calculate the probability that the geological conditions were favourable. **[5 marks]**

(d) To clarify the geological characteristics of the site, the oil company considers contracting a firm that performs seismic tests. The results of the tests are, however, uncertain: The test is 87% reliable for identifying correctly favourable geological conditions and 78% reliable in identifying correctly unfavourable geological conditions. To carry out the tests the firm will charge the oil company 15 million. Decide (on the basis of expected profits) whether the company should contract the firm to carry out the tests. **[5 marks]**

Question 4

(a) Consider a portfolio of two assets, whose unit return rates, denoted by the random variables ξ_1 and ξ_2, satisfy: $\mathbb{E}[\xi_1] = 1$, $\mathbb{E}[\xi_2] = 2$, $\mathbb{E}[(\xi_1 - 1)^2] = \mathbb{E}[(\xi_2 - 2)^2] = 1$ and $\mathbb{E}[(\xi_1 - 1)(\xi_2 - 2)] = -0.5$, where $\mathbb{E}(\cdot)$ is the expectation operator. Assume also that w_1 and w_2 are the portfolio weights which satisfy $w_1 + w_2 = 1$, $w_1 \geq 0$ and $w_2 \geq 0$. Let the random variable $\xi = w_1\xi_1 + w_2\xi_2$ denote the unit return of the portfolio. Find the value of γ in the interval $1 \leq \gamma \leq 2$ that minimises the portfolio variance subject to the constraint $\mathbb{E}[\xi] = \gamma$ and the corresponding values of the weights w_1 and w_2. **[5 marks]**

(b) Let X be the random variable representing the gains of a lottery which offers an amount x_1 with probability p and an amount x_2 with probability $1 - p$, where $x_1 < x_2$. Let $u(\cdot)$ be the utility function of a risk-averse person which satisfies:

$$u(\mathbb{E}[X]) \geq \mathbb{E}[u(X)] \quad \text{for all } p \in [0, 1]$$

in which $\mathbb{E}(\cdot)$ denotes the expectation operator. Explain by means of a diagram why the utility function $u(x)$, $x_1 \leq x \leq x_2$, is concave. **[5 marks]**

(c) You are faced with a decision as to whether to purchase insurance on a hired item against damage. You estimate that the probability that the item will not be damaged is 90%. If the item is damaged the amount of damage X is £100 with probability 9% and £1000 with probability 1%. Assume that insurance is offered with £200 excess (i.e. if the amount of damage is X you have to pay $\min(X, 200)$).

(i) Assuming that you are risk-neutral (and therefore you take decisions by minimizing expected loss) determine the maximum value of insurance you would be willing to pay. **[5 marks]**

(ii) Assuming that your utility reduction to a loss of £x is $(x/100)^2$ calculate the new maximum value of insurance you would be willing to pay and comment on any differences from your result in part (c)(i). **[5 marks]**

Table of Normal Distribution cdf $\Phi(z)$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2} d\xi$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999